Math 70100: Functions of a Real Variable I
Homework 10, due Wednesday, November 19th.

Remark on conventions: By convention, a *measurable set* in $\mathbb{R}$ is a Lebesgue measurable set. If $E \subset \mathbb{R}$ is (Lebesgue) measurable and $f : E \to \mathbb{R}$ is a function, then by convention f is *measurable* if it is $(\mathcal{L}, \mathcal{B})$-measurable, i.e., if the preimage of every Borel measurable set in $\mathbb{R}$ is a Lebesgue measurable subset of E.

1. *(Modified from Royden-Fitzpatrick, §2.7 # 46)* Let X and Y be topological spaces. Prove that every continuous function $f : X \to Y$ is Borel measurable. That is, prove that the preimage of a Borel set in Y is a Borel set in X. (*Hint:* The collection of sets E for which $f^{-1}(E)$ is Borel is a σ-algebra containing the open sets.)

Solution: We follow the hint. Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be the Borel σ-algebras on X and Y, respectively. Define

$$\Sigma = \{E \subset Y \ : \ f^{-1}(E) \text{ is Borel}\}.$$

Our claim that $f^{-1}(E)$ is Borel whenever E is Borel, then becomes the statement that $\mathcal{B}_Y \subset \Sigma$.

First we claim that Σ is a σ-algebra. We need to show it is closed under compliments and countable unions. Suppose $E \in \Sigma$. Then $f^{-1}(E) \in \mathcal{B}_X$ is Borel. Observe that

$$f^{-1}(E^c) = f^{-1}(E)^c \in \mathcal{B}_X.$$

Therefore, $E^c \in \Sigma$. Now suppose $\{E_i\} \subset \Sigma$ is a countable subset. Then,

$$f^{-1}\big(\bigcup_i E_i\big) = \bigcup_i f^{-1}(E_i).$$

Since each $f^{-1}(E_i) \in \mathcal{B}_X$, we know that $f^{-1}\big(\bigcup_i E_i\big) \in \mathcal{B}_X$. Thus, $\bigcup_i E_i \in \Sigma$. This finishes the proof that Σ is a σ-algebra.

Let $U \subset Y$ be open. Because f is continuous, we know $f^{-1}(U)$ is open and hence $f^{-1}(U) \in \mathcal{B}_X$. We conclude that $U \in \Sigma$ for every $U \subset Y$ open.

In summary, Σ is a σ-algebra for Y containing the open sets. By definition $\mathcal{B}_Y$ is the smallest σ-algebra containing the open subsets of Y. Therefore $\mathcal{B}_X \subset \Sigma$. Thus, the preimage of every Borel set is Borel.

2. Let $E \subset \mathbb{R}$ be Lebesgue measurable and $f : E \to \mathbb{R}$ be a function. Let $g : E \to \mathbb{R}$ be another function. We say $f = g$ *almost everywhere* if there is a subset $Z \subset E$ of Lebesgue measure zero so that $f(x) = g(x)$ for all $x \in E \setminus Z$. Show that if $f = g$ almost everywhere and g is measurable, then f is measurable.

Solution: Let $Z \subset E$ be a subset of Lebesgue measure zero so that $f(x) = g(x)$ for $x \in E \setminus Z$. To prove that f is measurable, it suffices to prove that $f^{-1}(B)$ is Lebesgue measurable for every Borel set $B \subset \mathbb{R}$. Since g is measurable $g^{-1}(B)$ is measurable. Since $f = g$ on $E \setminus Z$, there is a subset $W \subset Z$ so that

$$f^{-1}(B) = \big(g^{-1}(B) \cap (E \setminus Z)\big) \cup W.$$

Since $W \subset Z$ and Lebesgue measure is compete, W is also Lebesgue measurable. From the expression above, we see $f^{-1}(B)$ is measurable as well.

3. Let $\{E_i \subset \mathbb{R}\}$ be a countable collection of measurable sets, and let $E = \bigcup_i E_i$. Let $f : E \to \mathbb{R}$ be a function. Show that f is measurable if and only if $f|_{E_i}$ is measurable for each i.

Solution: First suppose f is measurable. Fix i. We will show f_{E_i} is measurable. Let $B \subset \mathbb{R}$ be a Borel set. Since f is measurable, $f^{-1}(B)$ is measurable. Then $f|_{E_i}^{-1}(B) = E_i \cap f^{-1}(B)$ is measurable because both E_i and $f^{-1}(B)$ are measurable.

Now suppose each $f|_{E_i}$ is measurable. Let $B \subset \mathbb{R}$ be Borel. Observe that

$$f^{-1}(B) = \bigcup_i f|_{E_i}^{-1}(B)$$

is measurable because each $f|_{E_i}^{-1}(B)$ is measurable.

4. *(Royden-Fitzpatrick §3.1 #9)* Let f_n be a sequence of measurable functions defined on a measurable set $E \subset \mathbb{R}$. Define E_0 to be the set of points x in E at which $\{f_n(x)\}$ converges. Is the set E_0 measurable? If so, prove it. Otherwise, explain how to produce a counterexample.

Solution: The set E_0 is measurable. To prove it, we will use the fact that $\{f_n(x)\}$ converges if and only if the sequence is Cauchy. We observe that $\{f_n(x)\}$ is Cauchy if and only if for all natural numbers p, there is an $N \in \mathbb{N}$ so that for all $n, m \in \mathbb{N}$ with $n, m > N$ we have $|f_n(x) - f_m(x)| < \frac{1}{p}$.

Let $p, n, m \in \mathbb{N}$. Observe that because f_n and f_m are measurable, so is $f_n - f_m$. Therefore, the set

$$A_{p,n,m} = \{x \in E \ : \ |f_n(x) - f_m(x)| < \frac{1}{p}\} = (f_n - f_m)^{-1}\big((-\frac{1}{p}, \frac{1}{p})\big)$$

is measurable. Observe that x is Cauchy if and only if it lies in the set

$$\bigcap_{p \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{n,m \in \mathbb{N};\ n,m > N} A_{p,n,m}.$$

Since a sequence converges if and only if it is Cauchy, this set is E_0, which is measurable because each $A_{p,n,m}$ is measurable, and because we are only using countable intersections and unions above.

5. *(Modified from Pugh: Chapter 6 # 54)*
Egoroff's theorem. Let $E \subset \mathbb{R}$ be a measurable set of finite Lebesgue measure, and let $\{f_n\}$ be a sequence of functions which converges almost everywhere (i.e., there is a set $Z \subset E$ of zero Lebesgue measure so that $\{f_n(x)\}$ converges when $x \in E \setminus Z$). Then for each $\epsilon > 0$ there is a measurable set $S \subset E$ with $\lambda(E \setminus S) < \epsilon$ such that $\{f_n(x)\}$ converges uniformly for $x \in S$.

Prove Egoroff's theorem by using the following steps. *Setup:* Let $E \subset \mathbb{R}$ be a measurable set of finite Lebesgue measure, and suppose $\{f_n : E \to \mathbb{R}\}$ converges almost everywhere. Thus, there is a zero set Z and a $f : E \setminus Z \to \mathbb{R}$ so that $\{f_n(x)\}$ converges to $f(x)$ for $x \in E \setminus Z$.

(a) For $k, \ell \in \mathbb{N}$, set

$$X(k, \ell) = \{x \in E \setminus Z \ : \ \forall n \geq k, \ |f_n(x) - f(x)| < 1/\ell\}.$$

Observe these sets are measurable. Show that for each ℓ, $E \setminus Z = \bigcup_k X(k, \ell)$.

Solution: The sets are measurable because they are intersections over integers $n \geq k$ of $(f_n - f)^{-1}\big((\frac{-1}{\ell}, \frac{1}{\ell})\big)$.

The second statement holds by definition. Fix ℓ and choose $x \in E \setminus Z$. Then since $\{f_n(x)\}$ converges to $f(x)$, there is an n so that $k \geq n$ implies $|f_n(x) - f(x)| < \frac{1}{\ell}$. So, $x \in X(k, \ell)$.

(b) Given $\epsilon > 0$, show that there is a sequence $\{k_\ell \in \mathbb{R} \ : \ \ell \in \mathbb{N}\}$ so that by defining $X_\ell = X(k_\ell, \ell)$, we have $\lambda(E \setminus Z \setminus X_\ell) < \frac{\epsilon}{2^\ell}$.

Solution: Fix ℓ and ϵ. Observe that $X(k, \ell) \subset X(k+1, \ell)$ for all $k \in \mathbb{N}$. Since $E \setminus Z$ is the union over k of these sets, by continuity of measure, we see that $\lambda\big(X(k, \ell)\big)$ increases monotonically in k to $\lambda(E \setminus Z)$. In particular because $E \setminus Z$ has finite measure, we can choose an k_ℓ so that

$$\lambda\big(E \setminus Z \setminus X(k_\ell, \ell)\big) = \lambda(E \setminus Z) - \lambda\big(X(k_\ell, \ell)\big) < \frac{\epsilon}{2^\ell}.$$

(c) Let $X = \bigcap_\ell X_\ell$. Show that $\lambda\big(E \setminus X\big) < \epsilon$ and that $\{f_n|_X\}$ converges uniformly to $f|_X$. (Don't give two meanings to ϵ!)

Solution: Since Z has measure zero, $\lambda(E \setminus Z \setminus X_\ell) = \lambda(E \setminus X_\ell)$. These quantities are less than $\epsilon/2^\ell$. So,

$$\sum_\ell \lambda\big(\bigcup_\ell E \setminus X_\ell\big) < \epsilon.$$

Observe that

$$E \setminus X = E \setminus \bigcap_\ell X_\ell = \bigcup_\ell (E \setminus X_\ell),$$

so countable subadditivity implies that $\lambda(E \setminus X) < \epsilon$.

Now we claim that $\{f_n|_X\}$ converges uniformly to $f|_X$. Choose $\eta > 0$. Then we can select an ℓ so that $\frac{1}{\ell} < \eta$. To verify uniform convergence on X, it suffices to verify that

$$x \in X \text{ and } n \geq k_\ell \text{ implies } |f_n(x) - f(x)| < \frac{1}{\ell}.$$

But this is true because $x \in X \subset X_\ell = X(k_\ell, \ell)$, and the above is true for $x \in X(k_\ell, \ell)$ by definition.

6. Show that Egoroff's theorem is not necessarily true when E has infinite measure.

Solution: Define $f_n(x) = (1 - \frac{1}{n})x$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Clearly f converges everywhere to $f(x) = x$. On the other hand if $\epsilon > 0$ and $S \subset \mathbb{R}$ is any subset with $\lambda(\mathbb{R} \setminus S) < \epsilon$, we would have $\lambda(S) = \infty$ and we conclude that S is unbounded. Let $\{x_k \in S\}$ be a sequence tending to $\pm\infty$. Then it can not be true that $f_n \to f$ uniformly on S. Indeed suppose the convergence was uniform on S. Then there would be an N so that $x \in S$ and $n > N$ implies $|f_n(x) - f(x)| < 1$. But, fixing some $n > N$, we see that

$$|f_n(x) - f(x)| = \frac{1}{n}|x|.$$

In particular,

$$\lim_{k\to\infty} |f_n(x_k) - f(x_k)| = \lim_{k\to\infty} \frac{1}{n}|x_k| = \infty,$$

because the sequence $\{x_k\}$ is unbounded. On the other hand, because each $x_k \in S$, these numbers should all be bounded by 1. This is a contradiction.

7. *(Pugh, Chapter 6 #50)* Construct a monotone function $f : [0, 1] \to \mathbb{R}$ whose discontinuity set is exactly $\mathbb{Q} \cap [0, 1]$ or show that such a function can not exist. Prove your answer is correct.

Solution: (The distribution function for a countable weighted sum of point mass measures works! Here we use $\mu = \sum_{r\in\mathbb{Q}} \alpha(r)\delta_r$, where δ_r is the measure defined by $\delta(A) = 1$ if $r \in A$ and $\delta(A) = 0$ if $r \notin A$.)

Such a function can exist. Recall that $\mathbb{Q} \cap [0, 1]$ is countable. For each $r \in \mathbb{Q} \cap [0, 1]$, choose a real number $\alpha(r) > 0$ so that

$$\sum_{r\in\mathbb{Q}\cap[0,1]} \alpha(r) < \infty.$$

Now define a function

$$f : [0, 1] \to \mathbb{R}; \quad x \mapsto \sum_{r\in\mathbb{Q}\cap[0,1];\, r<x} \alpha(r).$$

Note that this converges because any sum over a subsequence of a convergent series also converges. This function is monotone, because if $x < y$, then

$$f(y) - f(x) = \sum_{r\in[x,y)\cap\mathbb{Q}} \alpha(r) > 0. \tag{1}$$

It is discontinuous at each $x \in \mathbb{Q}$ because if $y > x$, then

$$f(x) - f(y) > \alpha(x) > 0.$$

Now we will show it is continuous when $x \notin \mathbb{Q}$. Choose an $\epsilon > 0$. We will first verify that it is right-continuous at x. Because the series converges, there is a finite set $F \subset \mathbb{Q} \cap [0,1]$ so that

$$\sum_{r \in (\mathbb{Q} \cap [0,1]) \smallsetminus F} \alpha(r) < \epsilon.$$

Observe that

$$\bigcap_{y > 0} \mathbb{Q} \cap [x, y) = \emptyset.$$

So, there is a $y > x$ so that $\mathbb{Q} \cap [x, y) \subset (\mathbb{Q} \cap [0,1]) \smallsetminus F$. Thus, $f(y) - f(x) < \epsilon$ by equation 1, which verifies right-continuity at x. The same argument proves that f is left-continuous at x, because $\bigcap_{y<x} \mathbb{Q} \cap [y, x) = \emptyset$.