Math 70100: Functions of a Real Variable I
Homework 10, due Wednesday, November 19th.

Remark on conventions: By convention, a *measurable set* in $\mathbb{R}$ is a Lebesgue measurable set. If $E \subset \mathbb{R}$ is (Lebesgue) measurable and $f : E \to \mathbb{R}$ is a function, then by convention f is *measurable* if it is $(\mathcal{L}, \mathcal{B})$-measurable, i.e., if the preimage of every Borel measurable set in $\mathbb{R}$ is a Lebesgue measurable subset of E.

1. *(Modified from Royden-Fitzpatrick, §2.7 # 46)* Let X and Y be topological spaces. Prove that every continuous function $f : X \to Y$ is Borel measurable. That is, prove that the preimage of a Borel set in Y is a Borel set in X. (*Hint:* The collection of sets E for which $f^{-1}(E)$ is Borel is a σ-algebra containing the open sets.)

2. Let $E \subset \mathbb{R}$ be Lebesgue measurable and $f : E \to \mathbb{R}$ be a function. Let $g : E \to \mathbb{R}$ be another function. We say $f = g$ *almost everywhere* if there is a subset $Z \subset E$ of Lebesgue measure zero so that $f(x) = g(x)$ for all $x \in E \setminus Z$. Show that if $f = g$ almost everywhere and g is measurable, then f is measurable.

3. Let $\{E_i \subset \mathbb{R}\}$ be a countable collection of measurable sets, and let $E = \bigcup_i E_i$. Let $f : E \to \mathbb{R}$ be a function. Show that f is measurable if and only if $f|_{E_i}$ is measurable for each i.

4. *(Royden-Fitzpatrick §3.1 #9)* Let f_n be a sequence of measurable functions defined on a measurable set $E \subset \mathbb{R}$. Define E_0 to be the set of points x in E at which $\{f_n(x)\}$ converges. Is the set E_0 measurable? If so, prove it. Otherwise, explain how to produce a counterexample.

5. *(Modified from Pugh: Chapter 6 # 54)*
Egoroff's theorem. Let $E \subset \mathbb{R}$ be a measurable set of finite Lebesgue measure, and let $\{f_n\}$ be a sequence of measurable functions which converges almost everywhere (i.e., there is a set $Z \subset E$ of zero Lebesgue measure so that $\{f_n(x)\}$ converges when $x \in E \setminus Z$). Then for each $\epsilon > 0$ there is a measurable set $S \subset E$ with $\lambda(E \setminus S) < \epsilon$ such that $\{f_n(x)\}$ converges uniformly for $x \in S$.

   Prove Egoroff's theorem by using the following steps. *Setup:* Let $E \subset \mathbb{R}$ be a measurable set of finite Lebesgue measure, and suppose $\{f_n : E \to \mathbb{R}\}$ is a sequence of measurable functions which converges almost everywhere. Thus, there is a zero set Z and a $f : E \setminus Z \to \mathbb{R}$ so that $\{f_n(x)\}$ converges to $f(x)$ for $x \in E \setminus Z$.

   (a) For $k, \ell \in \mathbb{N}$, set
   $$X(k, \ell) = \{x \in E \setminus Z \;:\; \forall n \geq k,\ |f_n(x) - f(x)| < 1/\ell\}.$$
   Observe these sets are measurable. Show that for each ℓ, $E \setminus Z = \bigcup_k X(k, \ell)$.

   (b) Given $\epsilon > 0$, show that there is a sequence $\{k_\ell \in \mathbb{R} \;:\; \ell \in \mathbb{N}\}$ so that by defining $X_\ell = X(k_\ell, \ell)$, we have $\lambda(E \setminus Z \setminus X_\ell) < \frac{\epsilon}{2^\ell}$.

   (c) Let $X = \bigcap_\ell X_\ell$. Show that $\lambda\big(E \setminus X\big) < \epsilon$ and that $\{f_n|_X\}$ converges uniformly to $f|_X$. (Don't give two meanings to ϵ!)

6. Show that Egoroff's theorem is not necessarily true when E has infinite measure.

7. *(Pugh, Chapter 6 #50)* Construct a monotone function $f : [0, 1] \to \mathbb{R}$ whose discontinuity set is exactly $\mathbb{Q} \cap [0, 1]$ or show that such a function can not exist. Prove your answer is correct.