Math 70100: Functions of a Real Variable I
Homework 1, due Wednesday, September 10th.

1. *(Based on Pugh 2.92)* Let X be a topological space. Recall that a *neighborhood* N of $x \in X$ is a subset $N \subset X$ so that there is an open set $U \subset X$ with $x \in U$ and $U \subset N$. A *boundary point* of a set $A \subset X$ is a point $x \in X$ so that every neighborhood of x intersects both A and $X \setminus A$. The boundary of A, ∂A, is the set of all boundary points of A.
   (a) Show that $\partial A = X \setminus \big(\mathrm{Int}(A) \cup \mathrm{Int}(X \setminus A)\big)$.
   (b) Explain why ∂A is closed.
   (c) Show that $\partial\partial A \subset \partial A$.
   (d) Show that $\partial\partial\partial A = \partial\partial A$.
   (e) Given an example which illustrates that $\partial\partial A$ may not equal ∂A.

2. *(Pugh 2.37)* Let C denote the vector space of continuous functions from $[0,1]$ to $\mathbb{R}$. This space can be endowed with the sup (or L^∞) norm,
$$|f| = \sup\ \{|f(x)|\ :\ x \in [0,1]\}$$
or the L^1 norm,
$$\|f\| = \int_0^1 |f(x)|\, dx.$$
Consider the identity map between id from $(C, |\cdot|)$ to $(C, \|\cdot\|)$.
   (a) Show that id is a continuous. (Thus it is a continuous linear bijection.)
   (b) Show that the inverse id^{-1} is not continuous.

3. *(Modified from Lang II.5.3a)* Let ℓ^1 be the set of all sequences $\alpha = \{a_n\}_{n\in\mathbb{N}}$ of real numbers such that $\sum_{n\in\mathbb{N}} |a_n|$ converges. Define
$$|\alpha| = \sum_{n\in\mathbb{N}} |a_n|.$$
   (a) Prove that $|\cdot|$ is a norm on ℓ^1.
   (b) Recall that a sequence $\{\alpha_n\}$ in a normed vector space is *Cauchy* if given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that $|\alpha_m - \alpha_n| < \epsilon$ for $m, n \geq N$. A normed vector space is *complete* if all Cauchy sequences converge. Show that ℓ^1 is complete with the norm $|\cdot|$.

4. *(Lang II.13)* The *diagonal* Δ is the set of all points (x,x).
   (a) Show that a space X is Hausdorff if and only if the diagonal is closed in $X \times X$.
   (b) Show that a product of Hausdorff spaces is Hausdorff.

5. *(Lang II.5.5c)* Let X be a metric space. For each $x \in X$, define the function f_x on X by $f_x(y) = d(x,y)$. Let $\|\cdot\|$ be the sup norm.
   (a) Show that $d(x,y) = \|f_x - f_y\|$.
   (b) Let a be a fixed element of X, and let $g_x = f_x - f_a$. Show that the map $x \mapsto g_x$ is a distance-preserving embedding of X into the normed space of bounded functions on X. (*Remark:* This shows that every metric space is isometric to a subset of a normed vector space.)

6. *(Lang II.5.8ab)* Let X be a topological space and E a vector space with norm $|\cdot|$. Let $M(X, E)$ denote the set of all maps from X to E. Let $B(X, E)$ denote the set of bounded maps from X to E endowed with the sup norm defined by $\|f\| = \sup\{|f(x)| \ : \ x \in X\}$. Let $BC(X, E) \subset B(X, E)$ be the set of bounded continuous maps.
   (a) Show that $BC(X, E)$ is closed in $B(X, E)$.
   (b) A *Banach space* is a complete normed vector space. Show that if E is a Banach space, then $B(X, E)$ is complete.
7. Let X be a topological space. Then, X is called *separable* if it has a countable base (or basis) for its topology. A set $A \subset X$ is *dense (in X)* if its closure $\bar{A} = X$.
   (a) *(Lang II.15)* Show that a separable space has a countable dense subset.
   (b) *(Lang II.16a)* Show that if X is a metric space and has a countable dense subset, then X is separable.
8. (Lang II.5.17). An *open covering* of a topological space X is a collection $\mathcal{U}$ of open sets so that $X = \bigcup_{U \in \mathcal{U}} U$. A *subcover* is a subset $\mathcal{V} \subset \mathcal{U}$ which is still a cover (i.e., $X = \bigcup_{V \in \mathcal{V}} V$).
   (a) Show that every open covering of a separable space has a countable subcovering.
   (b) Show that a disjoint collection of open sets in a separable space is countable.
   (c) Show that a base (or basis) for the topology of a separable space contains a countable base.