Math 70100: Functions of a Real Variable I: Final Exam Solutions

Friday, December 19th, 2014

Prof. Hooper

1. Let $X \subset \mathbb{R}$ be a measurable set.

   (a) (4 points) As a set, what is $L^1(X)$? (Recall, this is a collection of equivalence classes of real-valued functions...)

   Solution: As a set $L^1(X)$ is the collection of equivalence classes of integrable functions $X \to \mathbb{R}$, where two integrable functions are equivalent if they are equal almost everywhere (or satisfy $\int_X |f - g| = 0$).

   (b) (4 points) How is a topology placed on $L^1(X)$?

   Solution: The topology is the one induced by the L^1 norm:
   $$\|f\|_1 = \int_X |f|.$$

   (c) (8 points) Let $I : L^1(X) \to \mathbb{R}$ be the map given by $I(f) = \int_X f$. Explain why I is uniformly continuous using basic properties of the integral.

   Solution: The map I is actually 1-Lipschitz. Indeed, suppose $\epsilon > 0$, $f, g \in L^1$ and $\|f - g\|_1 \leq \epsilon$. This means that
   $$\int |f - g| \leq \epsilon.$$
   Observe that for every $x \in \mathbb{R}$, we have
   $$-|f(x) - g(x)| \leq f(x) - g(x) \leq |f(x) - g(x)|.$$
   Then by monotonicity of the integral, we have
   $$\int -|f - g| \leq \int (f - g) = \int |f - g|.$$
   By linearity of the integral, we can simplify this as
   $$-\int |f - g| \leq \int f - \int g \leq \int |f - g|.$$
   Therefore,
   $$\big|I(f) - I(g)\big| \leq \int |f - g| \leq \epsilon.$$
   We have shown that $\|f - g\|_1 \leq \epsilon$ implies $\big|I(f) - I(g)\big| \leq \epsilon$. This says I is continuous (and indeed 1-Lipschitz.)

(d) (8 points) Explain why (equivalence classes of) polynomials are dense in the space $L^1([0,1])$. (Give a proof based on results you know.)

Solution: It suffices to show that every open metric ball in $L^1([0,1])$ contains a polynomial. Fix $f \in L^1([0,1])$ and $\epsilon > 0$. Since continuous functions are dense in $L^1([0,1])$, we can find a continuous $g : [0,1] \to \mathbb{R}$ so that $\|f - g\|_1 < \frac{\epsilon}{2}$. Recall that the Weierstrass Approximation Theorem tells us that polynomials are uniformly dense in the space of continuous real-valued functions on $[0,1]$. So, we can find a polynomial p so that

$$\sup\{|g(x) - p(x)| \ : \ x \in [0,1]\} \leq \frac{\epsilon}{2}.$$

But this implies that $\|g - p\|_1 = \int_{[0,1]} |g - p| \leq \frac{\epsilon}{2}$. Then, by the triangle inequality,

$$\|f - p\|_1 \leq \|f - g\|_1 + \|g - p\|_1 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

2. (a) (6 points) Carefully state the dominated convergence theorem for $L^1(\mathbb{R})$.

Solution: Let $\{f_n : \mathbb{R} \to \mathbb{R}\}$ be a sequence of measurable functions which converge pointwise almost everywhere to a function $f : \mathbb{R} \to \mathbb{R}$. Suppose the sequence is dominated by an integrable function $g : \mathbb{R} \to \mathbb{R}$. (Here *dominated* means $|f_n(x)| \leq g(x)$ for a.e. $x \in \mathbb{R}$.) Then f is integrable and $\lim_{n\to\infty} \int |f_n - f| = 0$. (This also implies that $\lim_{n\to\infty} \int f_n = \int f$.)

(b) (6 points) State Lusin's theorem.

Solution: Let $f : [a,b] \to \mathbb{R}$ be a measurable function. Then, for any $\epsilon > 0$, there is a compact set $K \subset [a,b]$ so that $\lambda([a,b] \smallsetminus K) < \epsilon$ and the restriction $f|_K : K \to \mathbb{R}$ is continuous.

3. (a) (6 points) Give a sequence of continuous functions $f_n : \mathbb{R} \to [0,\infty)$ which converges to the zero function in the L^1-norm, but does not converge to the zero function uniformly.

Solution: The sequence of functions functions $f_n : \mathbb{R} \to [0,\infty)$ defined by

$$f_n(x) = \max\ \{0, 1 - n^2x^2\}$$

works because the L^1-norms are given by

$$\|f_n\|_1 = \int_{\frac{-1}{n}}^{\frac{1}{n}} 1 - n^2x^2\ dx = \left[x - \frac{n^2x^3}{3}\right]_{\frac{-1}{n}}^{\frac{1}{n}} = \frac{2}{n} - \frac{2}{3n} = \frac{4}{3n}.$$

Thus $\|f_n\|_1$ tends to zero as $n \to \infty$. Also, it does not converge to zero uniformly because $f_n(0) = 1$ for all n.

(b) (6 points) Give a sequence of continuous functions $g_n : [0,1] \to [0,\infty)$ which converges to the zero function pointwise (everywhere) but satisfies $\lim_{n\to\infty} \int g_n = \infty$.

Solution: Define $g_n : [0,1] \to [0,\infty)$ by

$$g_n(x) = \max\ \{0, n^3 x(1-nx)\}.$$

Observe that $g_n(x) = 0$ when $x \in \{0\} \cup [\frac{1}{n}, 1]$. Thus, for any x there is an N so that $n > N$ implies $g_n(x) = 0$.
The integrals are given by:

$$\int g_n = n^3 \int_0^{\frac{1}{n}} x - nx^2\ dx = n^3[\frac{x^2}{2} - \frac{nx^3}{3}]_0^{\frac{1}{n}} = n^3(\frac{1}{2n^2} - \frac{1}{3n^2}) = \frac{n}{6}.$$

So, $\int g_n \to \infty$ as $n \to \infty$.

4. (12 points) For $x \in \mathbb{R}$, the translation of $\mathbb{R}$ by x is the map $T_x : \mathbb{R} \to \mathbb{R}$ given by $T_x(a) = x + a$. Suppose $E \subset \mathbb{R}$ is a measurable set. Prove that there is a subset $F \subset E$ with $\lambda(E \smallsetminus F) = 0$ so that for every pair of points $x, y \in F$,
$$\lambda\big(E \cap T_{x-y}(E)\big) > 0.$$

Solution: Let E be a measurable set. Lebesgue's density theorem dictates that almost every point of E is a density point of E. Therefore, if F is the collection of density points of E, then $\lambda(F \smallsetminus E) = 0$.

Now choose $x, y \in F$. Since these are density points of E, we have

$$\lim_{r \to 0} \frac{\lambda\big(E \cap B_r(x)\big)}{\lambda\big(B_r(x)\big)} = 1 \quad \text{and} \quad \lim_{r \to 0} \frac{\lambda\big(E \cap B_r(y)\big)}{\lambda\big(B_r(y)\big)} = 1.$$

Therefore, we can choose an r so that

$$\frac{\lambda\big(E \cap B_r(x)\big)}{\lambda\big(B_r(x)\big)} > \frac{1}{2} \quad \text{and} \quad \frac{\lambda\big(E \cap B_r(y)\big)}{\lambda\big(B_r(y)\big)} > \frac{1}{2}.$$

Let $A = E \cap B_r(x)$ and $B = E \cap B_r(y)$. Since the measure of each ball is $2r$, we see that the inequalities above tell us that $\lambda(A) > r$ and $\lambda(B) > r$. Now observe that the translation T_{x-y} restricts to a measure preserving bijection from $B_r(y)$ to $B_r(x)$. In particular, letting $C = T_{x-y}(B)$, we see that $C \subset B_r(x)$ and $\lambda(C) > r$. The same statements also held for A: $A \subset B_r(x)$ and $\lambda(A) > r$.

We claim that A and C intersect in a set of positive measure. Assume to the contrary that $\lambda(A \cap C) = 0$. Then,
$$\lambda(C \smallsetminus A) = \lambda(C) - \lambda(A \cap C) = \lambda(C) > r.$$
On the other hand $C \smallsetminus A$ is disjoint from A and contained in $B_r(x)$, so by subadditivity of measure:

$$\lambda\big(B_r(x)\big) \geq \lambda(A) + \lambda(C \smallsetminus A) > r + r = 2r.$$

But this contradicts the observation that $\lambda\big(B_r(x)\big) = 2r$.

We have shown that $\lambda(A \cap C) > 0$. Now observe that $A \subset E$ and $C \subset T_{x-y}(E)$. Therefore monotonicity of measure tells us that

$$\lambda\big(E \cap T_{x-y}(E)\big) \geq \lambda(A \cap C) > 0.$$

5. (a) (6 points) State Egoroff's theorem in the context of a sequence of functions $f_n : [0,1] \to \mathbb{R}$ converging to a function $f : [0,1] \to \mathbb{R}$.

Solution: Let $\{f_n : [0,1] \to \mathbb{R}\}$ be a sequence of measurable functions which converges pointwise almost everywhere to a (necessarily measurable) function $f : [0,1] \to \mathbb{R}$. Then, for every $\epsilon > 0$, there is a measurable set $E \subset [0,1]$ with $\lambda([0,1] \smallsetminus E) > \epsilon$ so that f_n converges uniformly to f on E.

(b) (10 points) Prove that if the sequence of functions $\{f_n : [0,1] \to \mathbb{R}\}$ converges pointwise almost everywhere to the zero function, then there is a non-decreasing sequence of non-negative real numbers c_n so that $\lim_{n\to\infty} c_n = +\infty$ and so that $c_n f_n(x)$ converges for almost every $x \in [0,1]$.

Solution:

Preliminary remarks: Most people seemed to have a lot of trouble with this one. There are really three key observations:

- Egoroff's theorem gives uniform convergence on sets arbitrary close to full measure.
- If $f_n \to 0$ uniformly on a set A, then for any constant $c > 0$ and any $\epsilon > 0$ there is an N so that $n > N$ implies $c|f_n(x)| < \epsilon$ for $n > N$.
- You can do this for a sequences of sets A increasing to full measure, for an increasing sequence of constants c and a decreasing sequence of constant ϵ to make $c_n f_n \to 0$ on a set of full measure.

Formal Proof: Fix some natural number m. Egoroff's theorem gives a measurable subset A_m with measure greater than $1 - \frac{1}{m}$ so that $f_n \to 0$ uniformly. We can take these sets to be increasing in m (i.e., $A_m \subset A_{m+1}$), because if $f_n \to f$ uniformly on a finite collection of sets, then it also converges uniformly on their union.

By definition of uniform convergence, for each m, there is an $N = N(m)$ so that $n > N(m)$ implies

$$|f_n(x)| < \frac{1}{4^m} \quad \text{for all } x \in A_m. \tag{1}$$

Now let $\{c_n\}$ be the sequence of numbers defined by

$$c_n = \sup\big\{2^m \ : \ m \in \mathbb{N} \text{ and } n > N(m)\big\},$$

whenever this set is non-empty. Define $c_n = 0$ if the set is empty. Then by construction, $\{c_n\}$ is non-decreasing and $c_n \to \infty$.

Now let $A = \bigcup_m A_m$. Observe that by monotonicity of measure, $\lambda(A) = 1$. Consider an $x \in A$. We claim $c_n f_n(x) \to 0$. Choose $\epsilon > 0$. We will find an N so that $n > N$ implies $c_n|f_n(x)| < \epsilon$. Since $x \in A$, there is an M so that $x \in A_m$ for $m > M$. We can then choose an $m > M$ so that $\frac{1}{2^m} < \epsilon$. Let $N = N(m)$ as above. Fix $n > N$. Then there is a largest $m^* \geq m$ so that $n > N(m^*)$. So, by definition we have $c_n = 2^{m^*}$. Since $m^* \geq m$, we know $x \in A_{m^*}$ since $x \in A_m$. Thus equation 1 tells us that $|f_n(x)| < \frac{1}{4^{m^*}}$. Putting this together, we see

$$c_n|f_n(x)| \leq \frac{2^{m^*}}{4^{m^*}} = \frac{1}{2^{m^*}} \leq \frac{1}{2^m} < \epsilon,$$

which verifies that $c_n|f_n(x)| \to 0$.

6. (12 points) Recall that a function $f : [0,1] \to \mathbb{R}$ is L-Lipschitz if

$$\big|f(x) - f(y)\big| \leq L|x - y| \quad \text{for all } x, y \in [0,1].$$

Prove that if $f : [0,1] \to \mathbb{R}$ is L-Lipschitz and $E \subset [0,1]$ is measurable, then $f(E)$ is measurable and $\lambda\big(f(E)\big) \leq L\lambda(E)$. (*Hint:* Prove it for the case when E is an interval first.)

Solution: First suppose $E \subset [0,1]$ is an interval. Then because f is continuous $f(E)$ also an interval, and thus measurable. Now suppose that $\lambda \circ f(E) > Lf(E)$. Since $f(E)$ is an interval, there would be two points $a, b \in f(E)$ so that $|f(a) - f(b)| > Lf(E)$. But this violates the Lipschitz property, since a and b lie in the interval E and $|a - b| \leq \lambda(E)$.

Now suppose $E \subset [0,1]$ is a countable disjoint union of intervals. Then $E = \bigcup_n I_n$. Observe that $f(E) = \bigcup_n f(I_n)$, which shows $f(E)$ is a countable union of intervals and thus measurable. Then by countable subadditivity of measure and the prior paragraph,

$$\lambda \circ f(E) \leq \sum_n \lambda \circ f(I_n) \leq \sum_n L\lambda(I_n) = L\lambda(E).$$

Finally consider a general measurable $E \subset [0,1]$. We will show that for any $n \in \mathbb{N}$ there are measurable sets A_n and B_n so that $A_n \subset E \subset B_n$ and $\lambda(B) \leq \lambda(A) + \frac{1}{n}$. This implies the measurability of E. (This is because

$$\bigcup_n A_n \subset E \subset \bigcap_n B_n,$$

while the measures of the two sets on the right and left are equal. We have shown E is the union of the measurable set $\bigcup_n A_n$ and a zero set and hence E is measurable.)

Fix E and n as above. Since E is measurable and of finite measure, there is a compact set K and and an open set $U \subset \mathbb{R}$ so that $K \subset E \subset U$ and $\lambda(U) < \lambda(K) + \frac{1}{nL}$. Since U and $U \smallsetminus K$ are open sets, they are countable unions of open intervals. So, the previous paragraph tells us that

$$\lambda \circ f(U) \leq L\lambda(U) \quad \text{and} \quad \lambda \circ f(U \smallsetminus K) \leq L\lambda(U \smallsetminus K).$$

Now because $K \subset U$ and $\lambda(U) < \lambda(K) + \frac{1}{nL}$, we observe that $\lambda(U \smallsetminus K) \leq \frac{1}{Ln}$. Observing that $f(U) \smallsetminus f(K) \subset f(U \smallsetminus K)$, we obtain

$$\lambda\big(f(U) \smallsetminus f(K)\big) \leq \lambda \circ f(U \smallsetminus K) \leq L\lambda(U \smallsetminus K) \leq \frac{1}{n}.$$

Equivalently, we have $\lambda \circ f(U) \leq \lambda \circ f(K) + \frac{1}{n}$. We also have $f(K) \subset f(E) \subset f(U)$. From the previous paragraph, we see that E is therefore measurable. Also observe that $f(E) \subset f(U)$ while $\lambda(U) < \lambda(E) + \frac{1}{nL}$ since $\lambda(E) > \lambda(K)$. Therefore,

$$\lambda \circ f(E) \leq \lambda \circ f(U) \leq L\lambda(U) \leq L\big(\lambda(E) + \frac{1}{nL}\big) = L\lambda(E) + \frac{1}{n}.$$

This is true independent of n, so $\lambda \circ f(E) \leq L\lambda(E)$.

7. (12 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable. Prove that for each $\epsilon > 0$ there is a δ so that for all $E \subset \mathbb{R}$ measurable, $\lambda(E) < \delta$ implies $\int_E |f| < \epsilon$, where λ denotes Lebesgue measure on $\mathbb{R}$.

Solution: Fix $\epsilon > 0$. For $N \in \mathbb{N}$, let

$$A_N = \{x \in \mathbb{R} \;:\; |f(x)| \leq N\}.$$

Let $f_N = \chi_{A_N} \cdot f$. Observe that the sequence of functions $|f_N|$ is non-decreasing pointwise and converges to $|f|$ pointwise. The sequence is bounded from above by the integrable function $|f|$, so by the Monotone convergence theorem (or Dominated convergence theorem),

$$\int |f_N| \to \int |f|.$$

We conclude that there is an N so that

$$\int_{A_N} |f| = \int |f_N| > -\frac{\epsilon}{2} + \int |f|.$$

Equivalently, we have

$$\int_{\mathbb{R} \smallsetminus A_N} |f| \leq \frac{\epsilon}{2}.$$

Set $\delta = \frac{\epsilon}{2N}$. Let $E \subset \mathbb{R}$ be a measurable set with $\lambda(E) < \delta$. Observe $E \subset E \cup (\mathbb{R} \smallsetminus A_N) = (E \cap A_N) \cup (R \smallsetminus A_N)$. Using this and monotonicity of the integral, we see

$$\int_E |f| \leq \int_{E \cup (\mathbb{R} \smallsetminus A_N)} |f| = \int_{E \cap A_N} |f| + \int_{R \smallsetminus A_N} |f| < \frac{\epsilon}{2} + \int_{E \cap A_N} |f|.$$

Then because $|f| \leq N$ on $E \cap A_N$, we see that

$$\int_{E \cap A_N} |f| \leq \int_{E \cap A_N} N = N\lambda(E \cap A_N) \leq N\lambda(E) \leq \frac{\epsilon}{2}.$$

Combining the two equations above, we see that

$$\int_E |f| < \frac{\epsilon}{2} + \int_{E \cap A_N} |f| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$