TOPOLOGICAL DYNAMICS OF ROTATIONS

W. PATRICK HOOPER

The purpose of these notes is to give an alternate proof of the statement that forward orbits of irrational rotations are dense.

These notes are long primarily because they are detailed. We first describe the 1-torus (which essentially just a circle). Then we describe a basic fact about open sets in $\mathbb{R}$, and use it to derives related facts on the 1-torus. We also describe a basic result about the compliment of a forward orbit (Lemma 8). Given these basic ingredients, the proof that a forward orbit of an irrational rotation is dense is short.

I hope that the ingredients in the proof may be useful to you as well.

1. The 1-torus and the circle

The real numbers act by addition on the real line. This has the effect of translating points in the line. We will find it useful to use this construction to act on subsets $S \subset \mathbb{R}$. Namely, if $x \in \mathbb{R}$ and $S \subset \mathbb{R}$, we define

$$x + S = \{x + s \ : \ s \in S\}.$$

Geometrically, we are just translating the set S by x. Also we can add two subsets $A, B \subset \mathbb{R}$:

$$A + B = \{a + b \ : \ a \in A \text{ and } b \in B\}.$$

Note that $\mathbb{Z} + \mathbb{Z} = \mathbb{Z}$.

The 1-*dimensional torus* (or 1-*torus*) is the additive group $\mathbb{T} = \mathbb{R}/\mathbb{Z} = \{x + \mathbb{Z} \ : \ x \in \mathbb{R}\}$. Here, $x + \mathbb{Z}$ denotes a set:

$$x + \mathbb{Z} = \{x + n \ : \ n \in \mathbb{Z}\}.$$

This set is an abelian group with the operation of addition:

$$(x + \mathbb{Z}) + (y + \mathbb{Z}) = (x + y) + (\mathbb{Z} + \mathbb{Z}) = x + y + \mathbb{Z}.$$

The torus $\mathbb{T}$ inherits the quotient topology from $\mathbb{R}$. To understand this, observe that there is a natural projection

$$\pi : \mathbb{R} \to \mathbb{T} \ : \ x \mapsto x + \mathbb{Z}.$$

The *quotient topology* defines a subset $S \subset \mathbb{T}$ as open if its pre-image

$$\pi^{-1}(S) = \{x \in \mathbb{R} \ : \ \pi(x) \in S\}$$

is open. Similarly, $S \subset \mathbb{T}$ is closed if its pre-image $\pi^{-1}(S)$ is closed. The 1-torus is also a metric space, meaning there is a distance function $d_{\mathbb{T}} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ which satisfies the definition of a metric space. The distance on $\mathbb{T}$ is induced by the usual distance on $\mathbb{R}$, which is defined so that the distance between $x, y \in \mathbb{R}$ is just $|x - y|$. The distance between two points $x + \mathbb{Z}$ and $y + \mathbb{Z}$ in the 1-torus is simply the minimal distance between two points in the subsets of $\mathbb{R}$, $x + \mathbb{Z}$ and $y + \mathbb{Z}$, i.e.:

$$d_{\mathbb{T}}(x + \mathbb{Z}, y + \mathbb{Z}) = \min\{|x' - y'| \ : \ x' \in x + \mathbb{Z} \text{ and } y' \in y + \mathbb{Z}\}.$$

Date: February 2, 2013.

The 1-torus is essentially the same as a circle. Let S^1 denote the unit circle in the plane centered at the origin. This is a subset of the plane, and inherits a topology. A set $U \subset S^1$ is open if it is the intersection of an open subset of the plane and S^1. The unit circle also has a metric. The distance d_{S^1} between two points in S^1 is the length of the shortest arc between them.

The unit circle and the 1-torus are essentially the same from the point of view of topology. This is because the map

$$h : \mathbb{T} \to S^1; \quad x + \mathbb{Z} \mapsto \big(\cos(2\pi x), \sin(2\pi x)\big)$$

is a homeomorphism. The map also behaves nicely with the distance functions, because

$$d_{\mathbb{T}}(x + \mathbb{Z}, y + \mathbb{Z}) = \frac{1}{2\pi} d_{S^1}\big(h(x + \mathbb{Z}), h(y + \mathbb{Z})\big).$$

The advantage of working with the torus is that it has a natural additive group structure, which appears less natural from the point of view of the circle.

2. Open sets in $\mathbb{R}$

We will discuss a basic fact about open sets of real numbers. Namely, every open set is a union of disjoint open intervals with endpoints outside of the set.

Let $U \subset \mathbb{R}$ be open. Let $I = (a, b)$ be an open interval contained in U. Here $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ and $a < b$. We say I is *a maximal open interval in* U if neither of the endpoints of I lie in U. (This is trivially true when the endpoints are $\pm\infty$.)

Theorem 1. *Let $U \subset \mathbb{R}$ be open. The collection of maximal open intervals in U is disjoint. (That is, if two maximal open intervals intersect, then they are equal.) The set U is the union of the maximal open intervals in U.*

Proof. First we prove disjointness. Suppose I and J are maximal open intervals in U that intersect. We must prove that $I = J$. Assume to the contrary that $I \neq J$. Then, $I \not\subset J$ or $J \not\subset I$. Assume without loss of generality that $I \not\subset J$. Since I is an interval that contains a point in the interval J and is not contained in J, it must be that I contains one of the end points of I. But this is impossible since by definition of maximality the endpoints of J lie outside of U, while I must be a subset of U. This proves disjointness.

Let I be the union of all open intervals which contain x and are contained in U. It must be proved that I is non-empty, open, and an interval. We see I is non-empty, because by definition of open there is an open interval which contains x and is contained in U. To see I is open note that any union of open sets is open. To see that I is an interval, we will verify that I is path connected. We will show that there is a path in I which joins x to any other point in I. Let $y \in I$. Then by definition of I, there is an open interval $J \subset U$ which contains x and y. Also we have $J \subset I$. Since J is an interval, there is a path in J which joins x to y. Since $J \subset I$, this path also lies in I.

We claim that I is a maximal open interval. Let a be one of the endpoints. We need to prove that $a \notin U$. Suppose $a \in U$. Then $a \in \mathbb{R}$ and by definition of open set, there is an open interval J so that $a \in J$ and $J \subset U$. The open intervals intersect non-trivially in an open interval with endpoint a, so $I \cup J$ is another open interval which contains x and is contained in U. So by definition of I, we have $I \cup J \subset I$. But this is absurd, because $a \in J$ and $a \notin I$. $\square$

3. Open sets in the 1-torus (or circle)

Let $I=(a,b)\subset\mathbb{R}$ be an open interval with length $b-a$ satisfying $0<b-a\leq 1$. We'll say that an *open arc* is a set of the form $A=\pi(I)$ where $\pi:\mathbb{R}\to\mathbb{T}$ is the natural projection. The *length* of A is $\ell(A)=b-a$. We call $\pi(a)$ the *left endpoint* of A and $\pi(b)$ the *right endpoint* of A. These points coincide precisely when the length of the arc is one.

Let $U\subset\mathbb{T}$ be an open set. We call an open arc $A\subset U$ *maximal in* U if neither endpoint lies in U.

Proposition 2. *Let $U\subset\mathbb{T}$ be an open set which is non-empty and not equal to $\mathbb{T}$. Let $\tilde{U}=\pi^{-1}(U)\subset R$. Let $I\subset\mathbb{R}$ be an open interval. Then I is a maximal open interval in U if and only if $\pi(I)$ is a maximal open arc in U.*

This is simply because checking if the endpoints of I lie in $\tilde{U}$ is equivalent to checking if the endpoints of $\pi(I)$ lie in U.

Corollary 3. *Let $U\subset\mathbb{T}$ be an open set which is non-empty and not equal to $\mathbb{T}$. Then the collection of all maximal open arcs in U is disjoint. The set U equals the union of all maximal open arcs in U.*

Proof. First we prove disjointness. Let A and A' are maximal open arcs in U, and suppose they are not disjoint. Then there is an $x\in\mathbb{R}$ so that $x+\mathbb{Z}\in A\cap A'$. Then there are open intervals I and I' in $\mathbb{R}$ containing x so that $\pi(I)=A$ and $\pi(I')=A'$. These intervals are maximal by the prior proposition, and they both contain x. So, $I=I'$ by the Theorem 1. Then also $A=A'$.

The statement about the union also follows from combining Theorem 1 with this proposition. Observe that $\tilde{U}=\pi^{-1}(U)$ is a union of maximal open intervals. So, U is the union of projections of these intervals, which are maximal by the proposition. ☐

Proposition 4. *Let $U\subset\mathbb{T}$ be an open set which is non-empty and not equal to $\mathbb{T}$. Let $\epsilon>0$. Then there are only finitely many open arcs A in U which are maximal and satisfy $\ell(A)\geq\epsilon$.*

Proof. By the corollary, the collection of all maximal open arcs in U is a disjoint collection. The total length of a collection of disjoint arcs in $\mathbb{T}$ cannot exceed 1 (the length of $\mathbb{T}$), so there can be no more than $\frac{1}{\epsilon}$ maximal open arcs in U. ☐

Corollary 5 (Longest arc)**.** *Let $U\subset\mathbb{T}$ be an open set which is non-empty and not equal to $\mathbb{T}$. Then there is an maximal open arc $A\subset U$ with the property that if A' is another open arc in U then $\ell(A')\leq\ell(A)$.*

Proof. Choose any maximal open arc $B\subset U$. Let $\epsilon=\ell(B)$. By the prior proposition, there are only finitely many maximal arcs whose length non-strictly exceeds ϵ. Choose the longest of these to be A. Since every open arc $A'\subset U$ is contained in a maximal open arc A'', we have $\ell(A')\leq\ell(A'')\leq\ell(A)$. ☐

4. Dynamics of rotations

A rotation of the torus is a map induced by translation on the real line. Namely, if $\alpha\in\mathbb{R}$, we define

$$R_\alpha:\mathbb{T}\to\mathbb{T};\quad x+\mathbb{Z}\mapsto\alpha+(x+\mathbb{Z})=x+\alpha+\mathbb{Z}.$$

Recall that a map is *continuous* if the inverse image of every open set is open. Since translations of $\mathbb{R}$ preserve the collection of open sets, it can be shown that a rotation is continuous. Observe that the inverse map R_α^{-1} is given by $R_{-\alpha}$. So, in fact R_α is a homeomorphism. Also translations preserve the metric in $\mathbb{R}$. It follows that rotations preserve the metric on $\mathbb{T}$.

We will be studying the dynamical system $(\mathbb{T}, R_\alpha)$. Powers of R_α are given by

$$R_\alpha^n(x + \mathbb{Z}) = x + n\alpha + \mathbb{Z},$$

for all $n \in \mathbb{Z}$.

Proposition 6. *If α is rational, then every point in $\mathbb{T}$ is periodic under R_α. More specifically, if $\alpha = \frac{p}{q}$ in lowest terms, then every point in $\mathbb{T}$ has least period q under R_α. If α is irrational, no point in $\mathbb{T}$ is periodic under R_α.*

Proof. It suffices to show that the following statements are equivalent for each $\alpha \in \mathbb{R}$ and each positive integer n.

(1) The point $0 + \mathbb{Z} \in \mathbb{T}$ is period n under R_α.
(2) The number $n\alpha$ is an integer.
(3) Every $x + \mathbb{Z} \in \mathbb{T}$ is period n under R_α.

The proposition is easily derived from the equivalence of these statements.

First we will show that (1) implies (2). Suppose $R_\alpha^n(0 + \mathbb{Z}) = 0 + \mathbb{Z}$. Observe that $R_\alpha^n(0 + \mathbb{Z}) = n\alpha + \mathbb{Z}$. Statement (1) then implies that $n\alpha + \mathbb{Z} = \mathbb{Z}$. Then in particular $n\alpha + 0 \in \mathbb{Z}$. So, $n\alpha \in \mathbb{Z}$, which proves statement (2) holds.

Now we will show that (2) implies (3). Suppose that $n\alpha \in \mathbb{Z}$. Pick any $x + \mathbb{Z} \in \mathbb{T}$. We have

$$R^n(x + \mathbb{Z}) = x + n\alpha + \mathbb{Z} = x + \mathbb{Z},$$

because $\mathbb{Z}$ is preserved by translation by an integer $(n\alpha)$. This proves statement (3) holds.

Statement (3) can be seen to imply (1) by setting $x = 0$. □

Recall that the forward orbit of a point $x \in X$ under a map $T : X \to X$ is

$$\mathcal{O}^+(x) = \{T^n(x) \ : \ n \geq 0\}.$$

Theorem 7. *If α is irrational, then every forward orbit of R_α is dense in $\mathbb{T}$.*

To prove the theorem, we use two ingredients. First, we give a lemma about forward orbits.

Lemma 8. *Let (X, T) be a dynamical system. If $S \subset X$ is disjoint from $\mathcal{O}^+(x)$, then so is its preimage $T^{-1}(S)$.*

Proof. We prove the contrapositive. Suppose that $T^{-1}(S) \cap \mathcal{O}^+(x) \neq \emptyset$. Then, there is a $T^n(x) \in T^{-1}(S)$. By definition,

$$T^{-1}(S) = \{y \in S \ : \ T(y) \in S\}.$$

Therefore, $T\big(T^n(x)\big) \in S$. But $T\big(T^n(x)\big) = T^{n+1}(x)$, so we see that $\mathcal{O}^+(x)$ is not disjoint from S. □

Second, we give a simple fact about the action of rotations on arcs.

Proposition 9. *Let $A \subset \mathbb{T}$ be an open arc, and let $a + \mathbb{Z} \in \mathbb{T}$ be the left endpoint of A. Then for all $\alpha \in \mathbb{R}$ and all $n \in \mathbb{Z}$, $R_\alpha^n(A)$ is an open arc with the same length as A, and the left endpoint of $R_\alpha^n(A)$ is given by $R_\alpha^n(a + \mathbb{Z})$.*

Proof. By definition of open arc, there is an open interval $I = (a, b) \subset \mathbb{R}$ with length less than one so that $A = \pi(I)$. Then, by definition of the rotation as projection of translation, we have

$$R_\alpha^n(A) = \pi(n\alpha + I).$$

So the length of $R_\alpha^n(A)$ is the length of $n\alpha + I$ which is the same as I and A. Also the left endpoint of $n\alpha + I$ is $n\alpha + a$, and $\pi(n\alpha + a) = R_\alpha^n(a)$. □

Proof of Theorem 7. We will prove the contrapositive. Suppose there is an $x + \mathbb{Z} \in \mathbb{T}$ whose forward orbit

$$\mathcal{O}^+(x + \mathbb{Z}) = \{R_\alpha^n(x + \mathbb{Z}) \ : \ n \geq 0\}$$

is not dense. We will prove that α is rational.

Let S denote the closure of $\mathcal{O}^+(x + \mathbb{Z})$. By hypothesis $S \neq \mathbb{T}$. Let $U = \mathbb{T} \smallsetminus S$. This is an open set which is disjoint from the forward orbit $\mathcal{O}^+(x + \mathbb{Z})$. Let $A = \pi(I)$ be a maximal arc in U of the longest possible length in U. This exists by Corollary 5.

Let n be a positive integer. By induction application of Lemma 8, we see that $R_\alpha^{-n}(A)$ is disjoint from $\mathcal{O}^+(x + \mathbb{Z})$. Then, since $R_\alpha^{-n}(A)$ is open, it must also be disjoint from the closure of the orbit, S. So $R_\alpha^{-n}(A) \subset U$ for all n.

Observe that $R_\alpha^{-n}(A)$ has the same length as A. Because of our choice of A, any arc of equal length which lies in U must be maximal in U. See Corollary 5. There can be only finitely many maximal open arcs of this length in U by proposition 4. We conclude that there are integers m and n with $0 \leq m < n$ so that $R_\alpha^{-m}(A) = R_\alpha^{-n}(A)$. Now let $a + \mathbb{Z}$ be the left endpoint of A. By the prior proposition, the left endpoints of $R_\alpha^{-m}(A)$ and $R_\alpha^{-n}(A)$ are given by

$$a - m\alpha + \mathbb{Z} \quad \text{and} \quad a - n\alpha + \mathbb{Z},$$

respectively. These endpoints must coincide since the intervals are equal. So, $(a - m\alpha) - (a - n\alpha) \in \mathbb{Z}$. We see therefore that $(n - m)\alpha \in \mathbb{Z}$. Since $m \neq n$, we see that α is rational. □

The City College of New York, New York, NY, USA 10031
E-mail address: whooper@ccny.cuny.edu