Math-354: A collection of important results

Useful theorems from Calculus

A definition of continuity. A function f is continuous at a if for any sequence x_n converging to a, the sequence $f(x_n)$ converges to $f(a)$.

The Intermediate Value Theorem. Let $I = [a, b]$ be an interval, and suppose that $f : I \to \mathbb{R}$ is continuous. Suppose $y \in \mathbb{R}$ lies strictly between $f(a)$ and $f(b)$. (That is, either $f(a) < y < f(b)$ or $f(b) < y < f(a)$.) Then there is an $x \in [a, b]$ for which $f(x) = y$.

The Mean Value Theorem. Let $I = [a, b]$ be an interval, and suppose that $f : I \to \mathbb{R}$ is differentiable. Then, there is a $c \in (a, b)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The following can be viewed as a generalization of the Mean Value Theorem.

The Lagrange form of Taylor's theorem. Let $n > 0$ be an integer, and assume f is a C^{n+1} function to $\mathbb{R}$ defined in a neighborhood of $a \in \mathbb{R}$. Then, for x in a small neighborhood of a,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where $f^{(n)}$ denotes the n-th derivative of f and

$$R_n(x) = \frac{f^{(n+1)}(y)}{(n+1)!}(x-a)^{n+1}$$

for some y strictly between a and x.

Key theorems from sections 9.2-9.4

Modified Theorem 9.2.4. *(Triangle theorem)* Let I be an interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a continuous function. Let $p \in I$ be a fixed point.

- Let $y_1 \in I$ be a point for which $y_1 < p$. Assume that $x < f(x) < p$ whenever $y_1 < x < p$. Then, $(y_1, x] \subset \mathcal{B}(p, f)$.
- Let $y_2 \in I$ be a point for which $p < y_2$. Assume that $p < f(x) < x$ whenever $p < x < y_2$. Then, $[x, y_2) \subset \mathcal{B}(p, f)$.

Remark. I have not included the condition that f be increasing on the interval (y_1, y_2). This condition is stated in book's version, but the book never uses this condition in its proof of the theorem.

Theorem 9.3.7 *(Classification of fixed points)* Let $I \subset \mathbb{R}$ be an interval, and suppose $f : I \to I$ is a C^1 function. Suppose $p \in I$ is a period-n point.

- If $|(f^n)'(p)| < 1$ then p is an attracting period-n point.
- If $|(f^n)'(p)| > 1$ then p is a repelling period-n point.

The following follows from the definition of continuity above, with some work. Something similar to this was stated in class.

Continuity and convergence to periodic points. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose p is a period-n point and let $x \in \mathbb{R}$. Let $m > 0$ be an integer. Suppose $\lim_{j\to\infty} f^{jmn}(x) = p$. Then $x \in \mathcal{B}(p, f)$.

Schwarzian Min-Max Principle. Suppose $S_f(x) < 0$ wherever $f'(x) \neq 0$. Then the derivative f' has no positive local maximum, and no negative local minimum.

Restated Theorem 9.4.5 *(Schwarzian Basin Theorem)* Assume $f : \mathbb{R} \to \mathbb{R}$ is a C^3 function, and that $S_f(x) < 0$ whenever $f'(x) \neq 0$. Assume that p is an attracting period-n point. Then at least one of the following statements is true about the basin of p under f^n.

(1) $[p, \infty) \subset \mathcal{B}(p, f^n)$.

(1') $(-\infty, p] \subset \mathcal{B}(p, f^n)$.

(2) There is a point $x_c \geq p$ with $(f^n)'(x_c) = 0$ for which $[p, x_c] \subset \mathcal{B}(p, f^n)$.

(2') There is a point $x_c \leq p$ with $(f^n)'(x_c) = 0$ for which $[x_c, p] \subset \mathcal{B}(p, f^n)$.

It follows that at least one of the following statements is true about the basin of the orbit of p under f.

(1) $[p, \infty) \subset \mathcal{B}(\mathcal{O}_f^+(p), f)$.

(1') $(-\infty, p] \subset \mathcal{B}(\mathcal{O}_f^+(p), f)$.

(2) There is a point $y_c \in \mathcal{B}(\mathcal{O}_f^+(p), f)$ with $f'(y_c) = 0$.