Math-354: Practice for Midterm 2

For the midterm, you should: (This is just a rough guide. Other topics may appear on the midterm.)

1. You should be able to use transition graphs to prove that a function has points of specific periods. In particular, you should be able to apply Theorem 10.1.8 as needed.
2. You should know the ordering used in Sharkovski's theorem and you should be able to use this theorem.
3. You should be able to show that certain sets are dense (or not dense), and that certain maps are topologically transitive (or not topologically transitive). Likewise, you should be able to prove that certain sets are invariant or positively invariant.
4. You should know what the shift space on N symbols, Σ_N^+, is. You should also be able to use it to understand the dynamics of certain maps.
5. You should be able to prove that a map has sensitive dependence on initial conditions at a point.
6. You should know what a subshift of finite type is and be able to use it to understand certain maps.
7. You should be able to show that certain invariant sets are Cantor sets. (What are the three properties that define a Cantor set. Can you use symbolic dynamics to recognize it?)
8. You should understand what a subshifts of finite type is. You should be able to use it to understand piecewise expanding maps. You should be able to compute the number of period-n points of a shift of finite type.
9. Understand the basic definitions of chapter 10 (and the previous chaper). For example, *itinerary*, *invariant set*, positively invariant set, *dense*, *topologically transitive*, *sensitive dependence on initial conditions*, *expansive*, *Cantor set*, *piecewise expanding map*, *Markov partion* and *subshift of finite type.*

Practice problems: I will not provide explicit answers to these questions, but will be happy to answer questions during our review on Monday (or in office hours or in an appointment.) Please compare answers with other students in the class.

1. Do problems 10.6.1, 10.6.3 and 10.6.5 from the book.
2. Let f be a continuous function on the interval $[1, 7]$ such that $f(1) = 4$, $f(2) = 7$, $f(3) = 6$, $f(4) = 5$, $f(5) = 3$, $f(6) = 2$, $f(7) = 1$. Assume that the function is linear between these integers.
   (i) Label the intervals between the integers and determine the transition graph.
   (ii) For which n there is a period-n orbit? For any period that exists, determine the itinerary of the periodic point in terms of the intervals through which its orbits passes.

3. Let $T(x) = \begin{cases} 8x & \text{if } x \le \frac{1}{4}, \\ -8x+4 & \text{if } \frac{1}{4} < x \le \frac{1}{2} \\ 8x-4 & \text{if } \frac{1}{2} < x \le \frac{3}{4} \\ -8x+8 & \text{if } \frac{3}{4} \le 1 \end{cases}.$

   Describe the set of points x such that x, $T(x)$, $T^2(x)$ are all in $[0,1]$, that is, the set

$$C_2 = \left\{x \colon T^k(x) \in [0,1] \text{ for } 0 \le k \le 2\right\}.$$

   It is made up of how many intervals of what length?

4. Determine whether each of the following is true (T) or false (F). If the statement is false, explain what is wrong with the statement.

   (a) Suppose A and B are subsets of C. If A is dense in both C and $A \cap B$, then B is dense in C.

   (b) Suppose f is a continuous function on a closed interval I with I invariant under f. If f does not have a period-2 orbit on I, than it is possible that f has no periodic points on I.

   (c) Suppose we construct a point x in $[0,1]$ using its binary expansion by the following procedure:

$$x = 0.\underbrace{00000\ \ 0000100010\ldots 11111}_{\text{all strings of length 5}}\underbrace{0000000\ldots 1111111}_{\text{all strings of length 7}}\ldots.$$

   In other words we add all strings of length k for all odd $k \ge 5$. Then $\mathcal{O}_D^+(x)$ is dense in $[0,1]$, where D denotes the doubling map.

   (d) Let Σ_2^+ be the shift on the symbols $\{0,1\}$ and define $k : \Sigma_2^+ \to [0,1]$ by

$$k(s_0 s_1 s_2 \ldots) = \sum_{n=0}^{\infty} \frac{s_i}{2^{i+1}}.$$

   The map k is invertible.

5. Consider the numbers 1, 7, 2^{3114}, 48, 72, 73, 74. Use the Sharkovskiis Ordering to order them.

6. Consider the middle-half Cantor set $\mathcal{H}$ formed by starting with $[0,1]$ and deleting the middle half of each remaining subinterval instead of the middle third.

   (a) Let S_r be the set of points in $[0,1]$ that never map outside of the unit interval for the tent map T_r (slopes have absolute value r). For what value of r is $S_r = \mathcal{H}$?

   (b) Using the value of r found above, what intervals make up the set

$$\mathcal{H}_2 = \{x \in \mathbb{R} \ : \ T^j(x) \in [0,1] \text{ for } 0 \le j \le 2\}?$$

   (c) What numbers in $[0,1]$ belong to $\mathcal{H}$?

(d) Show that 1/20 is in H. What about 17/21?

(e) Show the the set $\widetilde{\mathcal{H}} = \mathcal{H} \cap [0, 0.1]$ is not dense in $[0, 0.1]$.

7. Binary and ternary expansions.

   (a) The following are written in base 2, convert them to base 10.

      i. 0.101_2.

      ii. 0.0101_2.

      iii. $0.101101101101\ldots_2$.

   (b) Give the ternary representations of two period-4 points of the tripling map, $f(x) = 3x \pmod 1$, that are not in the same orbit of the tripling map. Then convert these numbers to base 10 numbers.

8. Consider the function $f : [0,1] \to [0,1]$ given by the formula

$$f = \begin{cases} \frac{3}{10} + \frac{7}{3}x & \text{if } 0 \le x \le \frac{3}{10}, \\ \frac{7}{4} - \frac{5}{2}x & \text{if } \frac{3}{10} \le x \le \frac{7}{10}, \\ \frac{5}{3}x - \frac{7}{6} & \text{if } \frac{7}{10} \le x \le 1. \end{cases}$$

   (a) Find the stretching factor α.

   (b) Find a Markov partition for the map f (i.e., this partition should be a partition for the piecewise expanding map f that is also a Markov partition).

   (c) Draw the transition graph for f using your partition.

   (d) Does f satisfy the theorems conditions for it to be topologically transitive on $[0,1]$? Explain.

9. Suppose $f : [0,1] \to \mathbb{R}$ is a continuous map and $[0,1]$ is invariant with respect to f. Assume $I_L = [0, 0.5]$, $I_R = [0.5, 1]$, the shift space for f is Σ_2^+ and the order of the intervals along the real line is like that of the Tent map. Furthermore assume that for any integer $k > 0$, the length of the interval labeled with any k symbols $s_0 s_1 \ldots s_{k-1}$ satisfies

$$\text{length}(I_{s_0 s_1 \ldots s_{k-1}}) = \frac{\pi}{2^{k+1}}.$$

   (Recall $I_{s_0 s_1 \ldots s_{k-1}} = \{x \in [0,1] \ : \ f^j(x) \in I_{s_j} \text{ for all } 0 \le j \le k-1\}$.)

   (a) f has sensitive dependence on initial conditions on $[0,1]$.

   (b) Let $S = \{x \in [0,1] \ : \ x \text{ is a periodic point}\}$. Show that S is dense in $[0,1]$.