Math-354: Practice for Midterm 1

For the midterm, you should understand the following: (This is just a rough guide. Other topics may appear on the midterm.)

1. Understand the basic terms introduced in the course. For instance: *fixed point*, *period-n point*, *eventually period-n point*, and *basin of attraction*.
2. Understand how the number of fixed points of f^n for all n, relates to the number of period-n points, and the number of period-n orbits.
3. Understand how to iterate graphically, and how to tell if a fixed point is attracting (Theorem 9.2.4).
4. Understand the terms *L-stable*, *attracting*, *repelling*, *unstable*, and *semistable*. Also, understand how derivatives relate to stability of periodic points (Theorem 9.3.7).
5. Understand how to compute the Schwarzian derivative of a function, and what having a negative Schwarzian says about a function's dynamical properties.
6. Understand what tangential and period doubling bifurcations are, and basic facts about the derivatives which imply their existence.

Practice problems: (I will not provide explicit answers to these questions, but will be happy to answer questions during our review on Monday. Please compare answers with other students in the class.)

1. Consider the function $f(x) = \frac{5}{4}x - x^3$ defined on the whole real line. Without drawing a picture, find the fixed points and classify them as attracting, repelling, semistable or neither.
2. Let $f(x) = -\frac{3}{2}x^2 - \frac{1}{2}x + 1$. Consider the period-3 orbit $\{0, 1, -1\}$. Is this orbit attracting, repelling, semistable or neither?
3. Find the Schwarzian of the function $f(x) = \frac{3x - x^3}{2}$. Is it always negative? If not, find a point at which it is positive.
4. Let $f : [0, 1) \to [0, 1)$ be the tripling map $f(x) = 3x \pmod 1$. (Recall, (mod 1) means subtract the integer part.)
   (a) Determine if the following points are period-n, eventually period-n, or neither. Also determine n whenever applicable.
   $$a = \frac{1}{27}, \quad b = \frac{7}{10}, \quad c = \frac{1}{39}, \quad \text{and} \quad d = \frac{\pi}{6}.$$
   (b) Find a period-2 orbit of f. Is it attracting, repelling, semistable or neither?
   (c) Explain why every period-n point is repelling.

5. Consider the maps $G(x) = 4x(1-x)$ and $f(y) = y^2 - 2$.

   (a) Show that $y = C(x) = -4x + 2$ is a conjugacy from G to f.

   (b) Recall from the book that $x = h(z) = \sin^2(\frac{\pi z}{2})$ is a conjugacy from the tent map $T(z)$ to the logistic map $G(x)$. Find a conjugacy $y = K(z)$ from $T(z)$ to $f(y)$.

   (c) How many period-3 orbits does f have?

6. Suppose $f(x)$ is a function defined on $\mathbb{R}$ which has the special property that for every $k \geq 1$ the map f^k has $3^k - 2^k$ fixed points, that is, f has $3-2$ fixed points, f^2 has $3^2 - 2^2$ fixed points, etc. Complete the below given table :
(i) k,
(ii) number of fixed points of f^k,
(iii) how many of these are fixed by f^k but not period-k points for f,
(iv) number of period-k points for f,
(v) number of period-k orbits.
Restrict your consideration to $1 \leq k \leq 4$.

7. Let $f(x) = x^3$ and let $y = h(x) = 3x - 3$. Determine the map $g(y)$ such that h is a conjugacy from f to g. (Hint: Notice that $h^{-1}(y) = \dfrac{y+3}{3}$.)

8. Consider the function $F(x) = x^2$.

   (a) Determine all fixed points and classify each of them as attracting, repelling, semistable or neither.

   (b) Use the graphical method of iteration to determine the dynamics of all points. Describe the orbits of representative points using words as well as the plot (if possible, please use a ruler). Hint: Distinguish between the cases $x < -1$, $x = -1$, $-1 < x < 0$, $x = 0$, $0 < x < 1$, $x = 1$, $x > 1$.)

   (c) Does F have any period-2 points? Explain your answer!

9. Consider the family of maps $f_\mu(x) = \mu + x + \cos(x)$. Let $\mu_0 = -1$ and $x_0 = 0$.

   (a) Graph f_μ near x_0 for values of $\mu \in \{-0.9, -1.0, -1.1\}$. Answer the following questions using these graphs.

   (b) What type of bifurcation is occurring when $\mu_0 = -1$ and $x_0 = 0$?

   (c) How many fixed points appear within distance $1/2$ of x_0 in each case of $\mu \in \{-0.9, -1.0, -1.1\}$? (You may answer this graphically.)

   (d) Determine whether the points found above are attracting, repelling, semistable or neither.

10. Consider the family of maps $g_\mu(x) = \mu(x^2 - x)$. Let $\mu_0 = 1$.

   (a) Find a fixed point x_0 of g_{μ_0} with $g'_{\mu_0}(x_0) = -1$.

(b) Graph g_μ^2 for $\mu \in \{0.9, 1, 1.1\}$ near x_0. Use these graphs to answer the following questions.

(c) What type of bifurcation is occurring?

(d) Describe the stability of all fixed points and period-2 orbits relating to this bifurcation in the cases of $\mu \in \{0.9, 1, 1.1\}$. (You do not explicitly need to find the periodic points and the period-2 orbit.)