Math-354: Practice for the Final Exam

For the final, you should: (This is just a rough guide. Other topics may appear on the midterm.)

1. You should be able to do the things listed in the practice sheets for the previous two midterms. (I expect past material to take up about %60 of the exam.)
2. You should know the definitions of terms used in the course. You will not be asked to state definitions, but you may be asked to verify that a definition is satisfied.
3. You should be able to explicitly compute the Lyapunov and multipliers for points with simple orbits (e.g. eventually periodic points) and points within simple systems. You should be able to construct bounds for what the these numbers can be for simple systems (as in homework problem 11.3.3).
4. You should know what the ω-limit set is. You should be able to compute it in simple cases.
5. You should understand the terms *invariant set*, *attracting set*, *trapping region*, *attractor*, *chaotic invariant set*, *chaotic attracting set*, and *chaotic attractor*.
6. You should be able to prove that an invariant set is attracting using a trapping region.
7. You should be able to prove that an attracting set is an attractor using the ω-limit set.

Practice problems: I will not provide explicit answers to these questions, but will be happy to answer questions during our review on Monday (or in office hours or in an appointment.) Please compare answers with other students in the class.

1. Let $f(x) = rx(1-x)$ for $r = 0.5 + 17/2 \approx 3.06$. This map has a period-2 orbit which is approximately $x_1 = 0.581$ and $x_2 = 0.745$.
   (a) Determine the stability of the period-2 orbit using the values above.
   (b) What is the Lyapunov exponent of x_1?
   (c) Let $x_0 = f(0.5) \approx 0.765$. What is the Lyapunov exponent of x_0? Notice that 0.5 is the critical point.
2. Let $f(x) = \begin{cases} 3 + \frac{3}{2}(x-1) & \text{for } 0 \le x \le 1 \\ 3 - 2(x-1) & \text{for } 1 \le x \le 2 \\ 1 + \left(\frac{1+\sqrt{5}}{2}\right)(x-2) & \text{for } 2 \le x. \end{cases}$
   (a) Show that $[1,3]$ has a trapping region.
   (b) Show that f has a Markov partition on $[1,3]$. (Dont worry about the image of points outside $[1,3]$ like $x = 0$ or $x = 4$.)
   (c) Is f topologically transitive on $[1,3]$? If so, why?

(d) Does f have a chaotic attractor? If so, why?

(e) What estimate can you give for the Lyapunov exponents of orbits in $[1,3]$ which do not pass through the points 1 and 2 where the derivative does not exist? Give an estimate like $A \leq \ell(x_0; f) \leq B$ where A and B are specific values.

3. The function $f(x) = \cos x$ has a fixed point at approximately $p = 0.7391$. Approximate the Lyapunov exponent at this fixed point.

4. Let $f(x) = \dfrac{2}{3}x^3 - \dfrac{1}{2}x$.

   (a) Find the fixed points and determine their stability type as attracting or repelling.

   (b) Find the critical points, where $f'(x) = 0$.

   (c) Show the Schwarzian derivative of f is negative. Note:

$$S_f(x) = \frac{f'''(x)f'(x) - \frac{3}{2}f''(x)^2}{f'(x)^2}.$$

   (d) What is the ω-limit set of the critical points?

5. Assume that f is a differentiable function from $\mathbb{R}$ to $\mathbb{R}$, with $f(x_0) = x_1$, $f(x_1) = x_2$, $f(x_2) = x_0$, $f'(x_0) = 1/2$, $f'(x_1) = 2$, and $f'(x_2) = 1/3$.

   (a) Is the orbit $\mathcal{O}_f^+(x_0)$ attracting or repelling?

   (b) What is the Lyapunov exponent of x_0?

6. Consider the map given by

$$f(x) = \begin{cases} 4 + \frac{4}{3}x & \text{for } 0 \leq x \leq 3 \\ 8 - 2(x-3) & \text{for } 3 \leq x \leq 4 \\ 6 - \frac{3}{2}(x-4) & \text{for } 4 \leq x \leq 8 \end{cases}$$

on the interval $[0,8]$. Let

$$F(x) = \begin{cases} 4 + 2x & \text{for } -1 \leq x \leq 0 \\ f(x) & \text{for } 0 \leq x \leq 8 \\ 2(x-8) & \text{for } 8 \leq x \leq 9 \end{cases}$$

be the extension of $f(x)$ to the interval $[-1,9]$.

   (a) Sketch the graphs of f and F.

   (b) Find a partition for f and explain why it has the Markov property.

   (c) For what periods does f have a periodic orbit?

   (d) Show that F has a trapping region for the attracting set $[0,8]$.

(e) Does F have sensitive dependence on initial conditions on $[0, 8]$? You do not need to give a complete argument, but explain the basic reason why it has or does not have sensitive dependence on initial conditions.

(f) What can you say about the sign of Lyapunov exponents of points of the map F? You do not have to calculate an exact value, but what sign do they have to have?

(g) Is there a point x_0 whose forward limit set $\omega(x_0)$ is all of $[0, 8]$? Why or why or why not?

(h) Is the interval $[0, 8]$ a chaotic attractor for F? Explain what definition you are using for a chaotic attractor and why F satisfies the conditions of the definition.

7. Let $f(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 2x$.

   (a) Find the fixed points and classify them as attracting, repelling, or neither.

   (b) Use the graphical analysis to determine the dynamical behavior of all real initial conditions. Describe the orbits using words as well as by the plot.

8. Let $f : [1, 9] \to [1, 9]$ be a continuous function such that $f(1) = 5$, $f(2) = 9$, $f(3) = 8$, $f(4) = 7$, $f(5) = 6$, $f(6) = 4$, $f(7) = 3$, $f(8) = 2$, and $f(9) = 1$. Assume the the function is linear between these integers.

   (a) Label the intervals between the integers and give the transition graph.

   (b) For which n is there a period-n orbit? Give the Symbol sequence in terms of the intervals which will give each period that exists.

9. Let $f(y) = 1 - \frac{3}{4}y^2$ and $g(x) = 3x(1 - x)$. Show that $y = C(x) = 4x - 2$ is a conjugacy between $f(y)$ and $g(x)$.

10. Let $f(x) = \begin{cases} 5x + 4 & \text{for } x \leq -0.4 \\ -5x & \text{for } -0.4 \leq x \leq 0.4 \\ 5x - 4 & \text{for } 0.4 \leq x. \end{cases}$

   (a) Sketch the graph of f. Notice that $f(-1) = -1$, $f(-0.4) = 2$, $f(0.4) = -2$, and $f(1) = 1$.

   (b) Consider the sets

$$K_n = \{x \ : \ f(x) \in [-1, 1] \text{ for } 0 \leq j \leq n \} = \bigcap_{j=0}^{n} f^{-j}([-1, 1]).$$

   How many intervals do K_1 and K_2 contain and what is the length of each of these intervals?

   (c) Discuss the set $K = \bigcap_{n \geq 0} K_n$.