1. Let $f : [0,1) \to [0,1)$ be defined by $f(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{3} \\ 3x - \cancel{3}\,1 & \text{if } \frac{1}{3} \le x < \frac{2}{3} \\ 3x - \cancel{6}\,2 & \text{if } \frac{2}{3} \le x < 1. \end{cases}$

Consider the following subset of $[0,1)$.

$$U = \{x \in [0,1) \ : \ f^k(x) = 0 \text{ for some } k \ge 0.\}$$

Using symbolic dynamics, we have another equivalent definition of U.

$$U = \{\sum_{n=0}^{\infty} \frac{s_n}{3^{n+1}} \ : \text{ each } s_n \in \{0,1,2\} \text{ and there is a } k \text{ for which } s_k = 0.\}$$

Answer the following true or false questions. *(5 points each)*

(a) F The set U is a Cantor set.

U contains the interval $[0, \frac{1}{3}]$!

(b) T The set U is dense in $[0,1)$.

True: If $x = \sum_{n=0}^{\infty} \frac{s_n}{3^{n+1}}$ then $\sum_{n=0}^{N} \frac{s_n}{3^{n+1}} \in U$ and is within $\frac{1}{3^{N+2}}$ of x.

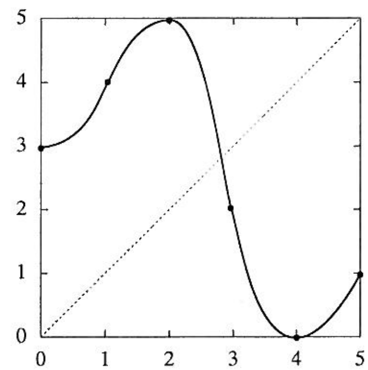
(c) T There is an irrational point $x \in [0,1)$ which is not in U.

For instance, ~~0.1211211121111~~ $(0.121121112111121111112\ldots)_3 \in U$.

(d) F The map f is topologically transitive.

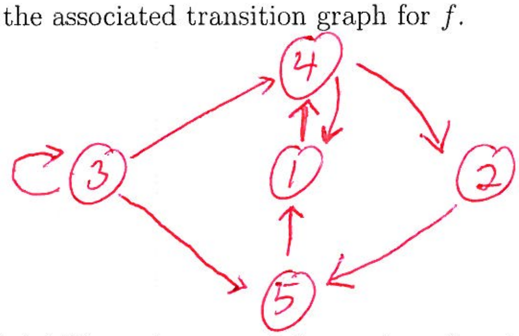
Suppose $\mathcal{O}^+_f(x)$ was dense then $\exists n$ such that $f^n(x) \in (0, \frac{1}{3})$. Then $f^{n+k}(x) = 0$ for all $k > 0$. So $\mathcal{O}^+_f(x)$ is finite and cannot be dense!

2. Let $f : [0,5] \to [0,5]$ be function which is graphed below.



Note that $f(0) = 3$, $f(1) = 4$, $f(2) = 5$, $f(3) = 2$, $f(4) = 0$ and $f(5) = 1$.

(a) *(5 points)* Let $I_1 = [0,1]$, $I_2 = [1,2]$, $I_3 = [2,3]$, $I_4 = [3,4]$ and $I_5 = [4,5]$. Draw the associated transition graph for f.



(b) *(5 points)* The point $x = 0$ is a point of period-6 under f. According to Sharkovskii's theorem, the map f must have points of which other periods?

All even periods and a fixed point.

(c) *(10 points)* For each period you mentioned in part (b) give an itinerary for a point of that period.

fixed: 3^∞

period-2: $(14)^\infty$

period-4: $(1425)^\infty$

period-2k for $k \geq 3$: $\left(1425(14)^{k-2}\right)^\infty$

3. Let $\Sigma_{\mathcal{G}}^{+}$ be the shift space of finite type whose transition matrix is $\mathbf{T} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

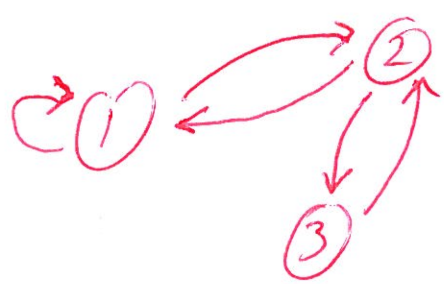
Powers of this matrix are given below.

$$\mathbf{T}^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \mathbf{T}^3 = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \quad \mathbf{T}^4 = \begin{bmatrix} 6 & 4 & 3 \\ 4 & 5 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\mathbf{T}^5 = \begin{bmatrix} 10 & 9 & 4 \\ 9 & 5 & 5 \\ 4 & 5 & 1 \end{bmatrix} \quad \mathbf{T}^6 = \begin{bmatrix} 19 & 14 & 9 \\ 14 & 14 & 5 \\ 9 & 5 & 5 \end{bmatrix} \quad \mathbf{T}^7 = \begin{bmatrix} 33 & 28 & 14 \\ 28 & 19 & 14 \\ 14 & 14 & 5 \end{bmatrix}$$

Assume the vertices of $\mathcal{G}$ are labeled by $\{1, 2, 3\}$.

(a) *(5 points)* Draw the transition graph associated to the matrix $\mathbf{T}$.



(b) *(5 points)* How many allowable strings are there of length 6 which begin at 3 and end with 1? (For instance, 323211 is such an allowable string of length 6.)

4. (Namely, 321111
321121
321211
323211)

(c) *(10 points)* Compute the number of periodic-n allowable sequences for the shift map on $\Sigma_{\mathcal{G}}^{+}$ for $1 \leq n \leq 6$.

n	# Fix σ^n	# ~~Period~~ Fixed-n points of lower period (sequences)	# Period-n sequences.
1	1	0	1
2	5	1	4
3	4	1	3
4	13	5	8
5	16	1	15
6	38	8	30

spacing.

4. Let $g : [0,1] \to [0,1]$ be the logistic map with parameter 2, $g(x) = 2x(1-x)$.

   Recall that g has *sensitive dependence on initial conditions* at x_0 if there is an $r > 0$ such that, for any $\delta > 0$, there is a $y_0 \in [0,1]$ with $|x_0 - y_0| < \delta$ and an integer $k \geq 0$ such that $|g^k(x_0) - g^k(y_0)| \geq r$.

   (a) *(10 points)* Show that for all $x \leq \frac{1}{4}$ we have $g(x) \geq \frac{3}{2}x$.

   For $x \leq \frac{1}{4}$, $1 - x \geq \frac{3}{4}$.

   So $g(x) \geq 2x\left(\frac{3}{4}\right) = \frac{3}{2}x$.

   (b) *(10 points)* Prove that g has sensitive dependence on initial conditions at $x_0 = 0$. (*Hint:* Use the definition and assume part (a).)

   Let $r = \frac{1}{4}$.

   Given δ, let $y_0 = \frac{\delta}{2}$. Then $|x_0 - y_0| = \frac{\delta}{2} < \delta$.

   We must show $\exists K$ such that $|g^K(y_0) - g^K(x_0)| \geq \frac{1}{4}$.

   Note: $g^K(x_0) = g^K(0) = 0$ since 0 is fixed.

   So $|g^K(y_0) - g^K(x_0)| = g^K(y_0)$.

   By contradiction, assume that $g^K(y_0) \leq \frac{1}{4}$ for all K.

   Then $g^K(y_0) \geq \left(\frac{3}{2}\right)^K y_0 = \left(\frac{3}{2}\right)^K \frac{\delta}{2}$.

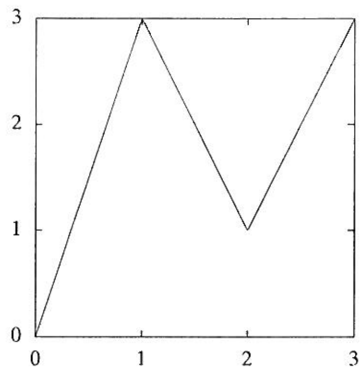
   Since $\lim_{K \to \infty} \left(\frac{3}{2}\right)^K \frac{\delta}{2} = +\infty$, we obtain

   a contradiction. to the assumption $g^K(y_0) \leq \frac{1}{4}$ for all K.

   Thus $\exists K \geq 0$ such that $g^K(y_0) \geq \frac{1}{4}$.

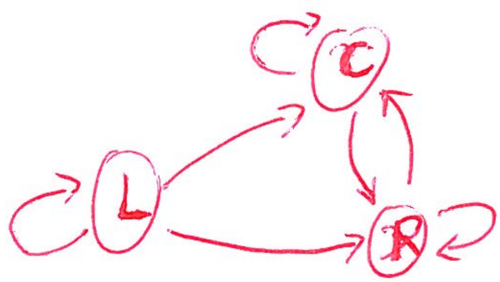
5. Let $f : [0,3] \to [0,3]$ be given by $f(x) = \begin{cases} 3x & \text{if } 0 \le x \le 1 \\ 5-2x & \text{if } 1 \le x \le 2 \\ 2x-3 & \text{if } 2 \le x \le 3. \end{cases}$

This function is graphed below.



Let $I_L = [0,1]$, $I_C = [1,2]$, $I_R = [2,3]$.

(a) *(5 points)* Draw the transition graph for f relative to the intervals above.



(b) *(5 points)* Is there a point $x \in [0,3]$ whose itinerary is $L(C)^\infty$? If so, find such a point. (Hint, you may want to find the point y whose itinerary is C^∞ first.)

The point y whose itinerary is C^∞ is fixed and lies in I_C. So $y = 5-2y$.
Thus $y = \frac{5}{3}$.
The point x lies in I_L and $f(x)=y$.
Therefore $3x = y$ and $x = \frac{5}{9}$.

Hint: Though I have given you a lot of space, your answer to parts (c) and (d) should be brief. Either quote a theorem, or give a brief heuristic explanation of your answer.

(c) *(5 points)* Does f have sensitive dependence ~~to~~ on initial conditions on $[0, 3]$? Why or why not?

Yes, f is a piecewise expanding map on the interval [0, 3]. So it has sdoic at every point in [0, 3].

(d) *(5 points)* Is f topologically transitive on $[0, 3]$? Why or why not?

No, the transition graph is irreducible. (Once an orbit lands in $I_C \cup I_R$, it cannot return to I_L.)