1. *(15 points)* For the following problems, circle the answer describing the stability of the fixed points in each question. You do not need to justify your answer.

   (a) Consider the function $f(x) = \frac{\pi}{2\sqrt{2}}\sin(x)$.
   The stability of the fixed point of f at $x = 0$ is best described as ___________.

   attracting   repelling   semistable   neither

   The stability of the fixed point of f at $x = \frac{\pi}{4}$ is best described as ___________.

   attracting   repelling   semistable   neither

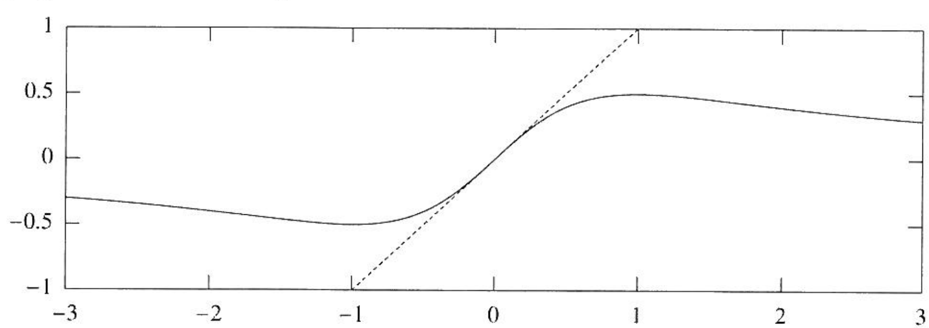
   (*Hint:* To two decimal places $\sqrt{2} = 1.41$ and $\pi = 3.14$.)

   $$f'(x) = \frac{\pi}{2\sqrt{2}}\cos(x)$$

   $$|f'(0)| = \left|\frac{\pi}{2\sqrt{2}}\right| \approx \frac{3.14}{2.82} > 1$$

   $$\left|f'\left(\frac{\pi}{4}\right)\right| = \frac{\pi}{2\sqrt{2}}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} < 1$$

   (b) The function $h(x) = \frac{x}{1+x^2}$ has derivative $h'(x) = \frac{1-x^2}{(1+x^2)^2}$. The function $h(x)$ is graphed with the diagonal below.



   The function $h(x)$ has a fixed point at $x = 0$ which is best described as ___________.

   attracting   repelling   semistable   neither

Apply triangle theorem.
Actually: $\mathcal{B}(0,h) = \mathbb{R}$!

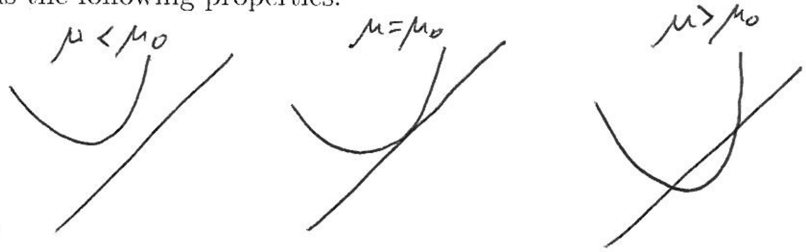
2. *(15 points)*
Suppose that $f_\mu(x) = f(x, \mu)$ is a C^3 function $\mathbb{R}^2 \to \mathbb{R}$. In addition, suppose the family of functions f_μ has the following properties.

(A) $f_0(0) = 0$.

(B) $f_0'(0) = 1$.

(C) $f_0''(0) = 4$.

(D) $\frac{d}{d\mu}[f_\mu(0)]\big|_{\mu=0} = -2$.



Hint: You may benefit by sketching the graph of f_μ for various values of μ.

(a) Which of the following bifurcations is occurring at $\mu = 0$ and $x = 0$? (Circle one.)

tangential bifurcation   period doubling bifurcation   neither

(b) The stability of the fixed point $x_0 = 0$ of f_0 is best described as ________.

attracting   repelling   semistable   neither

(see graph)

(c) Which of the following statements is most likely to be true?

i. The points $x_+ = \sqrt{\mu}$ and $x_- = -\sqrt{\mu}$ are fixed points of $f_\mu(x)$ for values of μ satisfying $0 \le \mu < 1$.

ii. The points $x_+ = \sqrt{-\mu}$ and $x_- = -\sqrt{-\mu}$ are fixed points for values of μ satisfying $-1 \le \mu < 0$.

iii. The point $x = 0$ is fixed for values of μ satisfying $-\frac{1}{2} < \mu < \frac{1}{2}$.

(see graph)

3. Consider the odd function $f_a(x) = x^3 - ax$ for values of a satisfying $0 < a < 1$. The first 3 derivatives of f_a are given by

$$f_a'(x) = 3x^2 - a, \quad f_a''(x) = 6x, \quad \text{and} \quad f_a'''(x) = 6.$$

The fixed points of f are the points $x_0 = 0$ and $x_\pm = \pm\sqrt{a+1}$.

(a) *(5 points)* Find the critical points of $f_a(x)$.

(b) *(10 points)* Compute the Schwarzian derivative of $f_a(x)$. Show that the Schwarzian is negative except at the critical points.

(a) Critical points are points where $f'(x) = 0$.

$$3x^2 - a = 0$$

$$3x^2 = a$$

$$x = \pm\sqrt{a/3}$$

(b) $$S_f(x) = \frac{f'(x)f'''(x) - \frac{3}{2}(f''(x))^2}{(f'(x))^2}$$

$$= \frac{6(3x^2-a) - \frac{3}{2}(6x)^2}{(3x^2-a)^2}$$

$$= \frac{-36x^2 - 6a}{(3x^2-a)^2} = \frac{-6(6x^2+a)}{(3x^2-a)^2}$$

So long as x is not a critical point, the denominator is positive. Also, the numerator is always negative, so $S_f(x) < 0$.

(c) *(5 points)* Prove that the points $x_\pm = \pm\sqrt{a+1}$ do not lie in the basin $\mathcal{B}(0, f_a)$. (*Hint: You may wish to recall the definition of the basin* $\mathcal{B}(0, f_a)$).

(d) *(10 points)* Assuming you have successfully answered the previous two parts to this question, explain why a critical point must lie in $\mathcal{B}(0, f_a)$.

(c) $\mathcal{B}(0, f_a) = \{x \in \mathbb{R} : \lim_{n\to\infty} |f^n(x)| = 0\}$.

But $x_\pm$ are fixed points so

$$\lim_{n\to\infty} |f^n(x_\pm)| = \lim_{n\to\infty} |x_\pm| \neq 0.$$

Thus $x_\pm \notin \mathcal{B}(0, f_a)$.

(d) Note that $|f'(0)| = |-a| = a < 1$, so 0 is an attracting fixed point. Also $S_f(x) < 0$ except at critical points. So, the Schwarzian basin theorem implies that either

(1) $[0, \infty) \subset \mathcal{B}(0, f_a)$,

(2) $(-\infty, 0] \subset \mathcal{B}(0, f_a)$, or

(3) There is a critical point in $\mathcal{B}(0, f_a)$.

But $\sqrt{a+1} \in [0, \infty)$ and $\sqrt{a+1} \notin \mathcal{B}(0, f_a)$ by (c) so (1) can not hold.

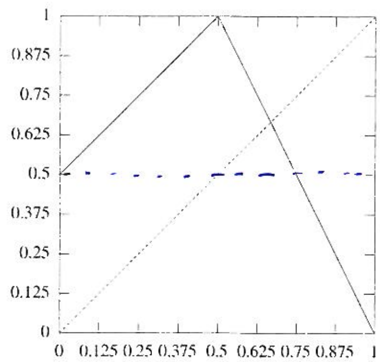
Also $-\sqrt{a+1} \in (-\infty, 0]$ and $-\sqrt{a+1} \notin \mathcal{B}(0, f_a)$ so (2) can not hold

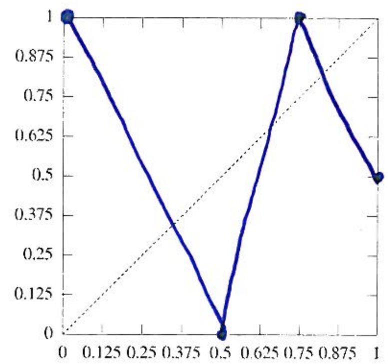
Thus (3) must be true.

4. *(20 points)* Let $h : [0,1] \to [0,1]$ be the continuous function defined by the following equation:

$$h(x) = \begin{cases} x + \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

(a) The graph of $h(x)$ is shown below together with the graph of the diagonal. Graph $h^2(x)$ in the second box below.





(b) Let a_n denote the number of fixed points of $h^n(x)$ for integers $n \geq 1$. The numbers a_n are recursively given by the following:

$$a_1 = 1, \quad a_2 = 3 \quad \text{and} \quad a_{n+2} = a_n + a_{n+1}.$$

Use this information to compute the number of period-n points for values of $n \in \{1, 2, 3, 4, 5, 6\}$.

n	$\#\mathrm{Fix}(h^n) = a_n$	# Points of lower period	# Points of period n.
1	1	0	1
2	3	1	2
3	4	1	3
4	7	3	4
5	11	1	10
6	18	6	12

5. *(20 points)* Let $f(x) = x^2$. Note that $f(x)$ has the following properties:

$$f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0, \quad \text{and} \quad f'(1) = 2.$$

Let $g(y) = 2y(1-y)$. Note that $g(y)$ has the following properties:

$$g(0) = 0, \quad g(\frac{1}{2}) = \frac{1}{2}, \quad g'(0) = 2, \quad \text{and} \quad g'(\frac{1}{2}) = 0.$$

(a) Find an affine conjugacy of the form $y = C(x) = mx + b$ from $f(x)$ to $g(y)$.

(b) Let $h(z) = 4z^2+z$. Is there a differentiable conjugacy from $f(x)$ to $h(z)$? If so, find the conjugacy. If not, explain why they are not differentiably conjugate.

(a) A conjugacy must send fixed points to fixed points.
A differentiable conjugacy must send a fixed point to a fixed point w/ the same derivative.

Therefore, $C(0) = \frac{1}{2} \ (= b)$

$C(1) = 0 \ (= m+b)$

So $b = \frac{1}{2}$ and $m = -\frac{1}{2}$. $C(x) = -\frac{1}{2}x + \frac{1}{2}$.

(b) Lets find the fixed points:

$h(z) = z$

iff $4z^2 + z = z \Rightarrow z = 0$.

So h has only one fixed point.
But a conjugacy must preserve the number of fixed points. So f and h are not conjugate!