Math-354: Metrics and Shift Spaces

A metric is a generalization of the notion of distance in a space. Let $x, y \in \mathbb{R}$. Then the usual distance between x and y is $|x - y|$. This is a metric on $\mathbb{R}$. The Euclidean distance, $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, between two points (x_1, y_1) and (x_2, y_2) in the plane is a metric on $\mathbb{R}^2$.

Definition. A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ such that for any $x, y, z \in X$

1. $d(x, y) \geq 0$.
2. $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) = d(y, x)$.
4. $d(x, z) \leq d(x, y) + d(y, z)$.

The last condition is called the triangle inequality. It can be interpreted as meaning that the trip from x to z is not longer than the trip from x through y to z. A *metric space* is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ is a metric on X.

Examples. In addition to the previous examples, here are some more examples of metrics. (You are encouraged to try to show these examples satisfy the definition of a metric.)

1. *(Discrete metrics.)* Let X be any set. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$$

2. *(The circle.)* Let $X = \{\theta \in \mathbb{R} \ : \ 0 \leq \theta < 2\pi\}$. Define

$$d(\alpha, \beta) = \min(|\alpha - \beta|, 2\pi - |\alpha - \beta|).$$

We can identify θ with the point $(\cos\theta, \sin\theta)$ on the unit circle in $\mathbb{R}^2$. From this point of view, $d(\alpha, \beta)$ is the length of the shortest arc of the circle joining $(\cos\alpha, \sin\alpha)$ to $(\cos\beta, \sin\beta)$.

3. *(Alternate metric on shift spaces.)* The following is related to (but not equal to) the metric on a shift space given in the book. Let $N \geq 2$ be an integer. Given $A = a_0 a_1 a_2 \ldots \in \Sigma_N^+$ and $B = b_0 b_1 b_2 \ldots \in \Sigma_N^+$ such that $A \neq B$ define $\Delta(A, B)$ to be the smallest integer $i \geq 0$ such that $a_i \neq b_i$. Such an integer exists because $A \neq B$. A metric on Σ_N^+ is given by

$$d(A, B) = \begin{cases} 0 & \text{if } A = B \\ \frac{1}{2^{\Delta(A,B)}} & \text{otherwise.} \end{cases}$$

For example $d(01010111\ldots, 01011110\ldots) = \frac{1}{2^4} = \frac{1}{16}$.
(*Remark:* This is still a metric if 2 is replaced by any number strictly larger than 1.)

Definition. Let (X, d) and (Y, d') be metric spaces, and let $f : X \to Y$ be any function. We say f is *continuous at a point* $x \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $y \in X$

$$d(x, y) < \delta \quad \text{implies} \quad d'\big(f(x), f(y)\big) < \epsilon.$$

We call $f : X \to Y$ *continuous* if it is continuous at all points of X.

Example. The shift map $\sigma : \Sigma_N^+ \to \Sigma_N^+$ is continuous when Σ_N^+ is given the metric d as in example (3) above.

Proof. Suppose $\epsilon > 0$. Then we can find a integer n such that $\frac{1}{2^{n-1}} < \epsilon$. Let $\delta = \frac{1}{2^n}$. We will show that for all $A, B \in \Sigma_N^+$ we have

$$d(x, y) < \delta = \frac{1}{2^n} \quad \text{implies} \quad d\big(\sigma(A), \sigma(B)\big) < \frac{1}{2^{n-1}} < \epsilon.$$

Let $A = a_0 a_1 a_2 \ldots$ and $B = b_0 b_1 b_2 \ldots$ be elements of Σ_N^+ such that $d(A, B) < \delta = \frac{1}{2^n}$. By definition of the metric d, we have either $A = B$ or $\Delta(A, B) \geq n + 1$. In either case, we have $a_i = b_i$ for all $i = 0, \ldots, n$. (So the first $n + 1$ symbols agree.) By definition of the shift map, we have

$$\sigma(A) = a_1 a_2 a_3 \ldots \quad \text{and} \quad \sigma(B) = b_1 b_2 b_3 \ldots .$$

Thus, the first n symbols of $\sigma(A)$ and $\sigma(B)$ agree. Thus, either $\sigma(A) = \sigma(B)$ or $\Delta(A, B) \geq n$. In either case, we have

$$d\big(\sigma(A), \sigma(B)\big) < \frac{1}{2^n} < \epsilon.$$

□

Exercise: Let d be the metric on Σ_2^+ defined in example (3) above. Let $h : \Sigma_2^+ \to [0, 1]$ be given by

$$h(a_0 a_1 a_2 \ldots) = \sum_{n=0}^{\infty} \frac{a_n}{2^{n+1}}.$$

Show that h is continuous.