Answer to 9.2.2 on Homework 1

9.2.2. Determine the dynamics for all points of $\mathbb{R}$ for the map $f(x) = \frac{3}{2}(x - x^3)$. (*As you will see, it is unreasonable to be able to explicitly determine the dynamics of all points!*

The fixed points of the map are solutions to the equation $f(x) = x$. We have

$$2\big(x - f(x)\big) = 3x^3 - x = 3x(x^2 - \frac{1}{3}).$$

So the fixed points are 0, $p_+ = \frac{\sqrt{3}}{3}$ and $p_- = \frac{-\sqrt{3}}{3}$.

We will now determine the types of these fixed points. We have $f'(x) = \frac{3}{2}(1 - 3x^2)$, so

$$f'(0) = \frac{3}{2} \quad \text{and} \quad f'(p_\pm) = 0.$$

We conclude that the fixed points p_+ and p_- are attracting, while 0 is repelling.

It is useful to consider the dynamics for positive and negative points with large magnitude. For polynomials of degree larger than 1, the orbits of such points diverge. For instance, when $|x|$ is very large, $|f(x)| \approx \frac{3}{2}|x|^3$. So, when $|x|$ is large, $|f^k(x)|$ increases very quickly.

We would like to determine the largest number q_+, for which it is no longer true that $|f(q_+)| > |q_+|$. By the intermediate value theorem, we must have $|f(q_+)| = |q_+|$. Note that $f(1) = 0$, so it must be that $q_+ > 1$. Now note that when $x > 1$ we have $f(x) < 0$. Therefore q_+ is a solution of the equation $x = -f(x)$. We have

$$2\big(f(x) + x\big) = 5x - 3x^3 = -3x(x^2 - \frac{5}{3}).$$

So the solutions to $x = -f(x)$ are $x = 0$ and $x = \frac{\pm\sqrt{5}}{\sqrt{3}}$. We conclude that $q_+ = \frac{\sqrt{5}}{\sqrt{3}}$. By the above argument, we know that if $x > q_+$, then $|f(x)| > x$. Similarly, if $x < -q_+$, then $|f(x)| > |x|$.

Let $q_- = \frac{-\sqrt{5}}{\sqrt{3}}$. We know $f(q_+) = q_-$ and $f(q_-) = q_+$. Therefore $\{q_+, q_-\}$ is a period-2 orbit. We compute

$$(f^2)'(q_\pm) = f'(q_+)f'(q_-) = (-6)^2 = 36.$$

So, this period-2 orbit is repelling.

Now we will consider the basin of attraction for the attracting fixed points. For points x satisfying $0 < x < p_+$ we have $x < f(x) < p_+$. Therefore, by the triangle theorem, the interval $(0, p_+]$ is contained in $\mathcal{B}(p_+, f)$. Now note that if $x \in [p_+, 1)$, then $f(x) \in (0, p_+]$. So in fact, $(0, 1) \subset \mathcal{B}(p_+, f)$. By a similar argument or by symmetry, $(-1, 0) \in \mathcal{B}(p_-, f)$.

We will see that the question of determining the dynamics for points x with $1 < |x| < q_+$ is quite a bit more delicate.

First let us try to compute the basin of attraction for the repelling fixed point 0. Of course, $0 \in \mathcal{B}(0, f)$. Notice that if $f(x) \in \mathcal{B}(0, f)$, then $x \in \mathcal{B}(0, f)$. Let $r_1 = 1$. Then $f(\pm r_1) = 0 \in \mathcal{B}(0, f)$, so $\pm r_1 \in \mathcal{B}(0, f)$. By looking at the graph, we can see that for any x satisfying $1 \leq x < q_+$, there is a unique y for which $f(y) = x$. Moreover, we have that $q_- < y < -x$. Similarly, if $q_- < x < -1$, then there is a unique y for which $f(y) = x$, and we have $x < y < q_+$. Therefore, there is an infinite sequence $r_1, r_2, r_3, \ldots$ of points for which $r_1 = 1$, $r_{i+1} > r_i$, and each $r_i < q_+$ which satisfy

$$f(r_{i+1}) = -r_i \quad \text{and} \quad f(-r_{i+1}) = r_i.$$

By induction we can conclude that each $r_i \in \mathcal{B}(0, f)$.

The figure on the last page may help explain what is going on.

The inverse function $f^{-1}(x)$ is well defined when $|x| \geq 1$. In the above, we are really studying the dynamics of f^{-1}, because inductively $f^{-1}(r_i) = -r_{i+1}$ and $f^{-1}(-r_i) = r_{i+1}$. We can conclude using the triangle theorem (theorem 9.2.4) that $[1, \infty) \subset \mathcal{B}(\mathcal{O}^+_{f^{-1}}(q_+), f^{-1})$. In particular, $\lim_{i \to \infty} r_i = q_+$.

The following facts can also be shown about the action of f on the intervals in between the points r_i.

1. $f\big((r_1, r_2)\big) = (p_-, 0)$ and $f\big((-r_2, -r_1)\big) = (0, p_+)$.
2. For $i > 1$, we have $f\big((r_i, r_{i+1})\big) = (-r_i, -r_{i-1})$ and $f\big((-r_{i+1}, -r_i)\big) = (r_{i-1}, r_i)$.

By induction, we can conclude that

$$f^i\big((r_i, r_{i+1})\big) = \begin{cases} (p_-, 0) & \text{if } i \text{ is odd} \\ (0, p_+) & \text{if } i \text{ is even.} \end{cases} \quad \text{and} \quad f^i\big((-r_{i+1}, -r_i)\big) = \begin{cases} (p_-, 0) & \text{if } i \text{ is even} \\ (0, p_+) & \text{if } i \text{ is odd.} \end{cases}$$

Finally, we recall that $(p_-, 0) \subset \mathcal{B}(p_-, f)$ and $(0, p_+) \subset \mathcal{B}(p_+, f)$, so this determines the dynamics of all points x:

- If $|x| > \frac{\sqrt{5}}{\sqrt{3}}$, then $\lim_{j \to \infty} |f^j(x)| = \infty$.
- If $|x| = \frac{\sqrt{5}}{\sqrt{3}}$, then x is a member of the period-2 orbit $\{q_+, q_-\}$.
- For each integer $k \geq 1$, we have $\pm r_k \in \mathcal{B}(0, f)$. Of course, $0 \in \mathcal{B}(0, f)$.
- We have $(r_i, r_{i+1}) \in \mathcal{B}(p_+, f)$ when i is even, and $(-r_{i+1}, r_i) \in \mathcal{B}(p_+, f)$ when i is odd for all $i \geq 1$. Also $(0, 1) \in \mathcal{B}(p_+, f)$.
- We have $(r_i, r_{i+1}) \in \mathcal{B}(p_-, f)$ when i is odd, and $(-r_{i+1}, r_i) \in \mathcal{B}(p_-, f)$ when i is even for all $i \geq 1$. Also $(-1, 0) \in \mathcal{B}(p_-, f)$.

This determines the dynamics of all points, because $\lim_{i \to \infty} r_i = q_+$. (It follows that $(1, q_+) = \bigcup i \geq 1 (r_i, r_{i+1})$.) Note that this description of the dynamics is not explicit in that we do not have an explicit formula for each r_i.

9.2.2. Plots of $y = \frac{3}{2}(x - x^3)$: