Answer to 9.1.7 on Homework 1

9.1.7. Suppose that q is odd and $q > 1$. Choose any $p \in \mathbb{Z}$ with $0 \leq p < q$. Then,

$$D(\frac{p}{q}) = \begin{cases} \frac{2p}{q} & \text{if } 2p < q \\ \frac{2p-q}{q} & \text{if } 2p \geq q. \end{cases}$$

In particular, the subset $S_q \subset [0,1)$ defined by

$$S_q = \{\frac{p}{q} \ : \ p \in \mathbb{Z} \text{ and } 0 \leq p < q\}$$

is *invariant under* D. That is, if $\frac{p}{q} \in S_q$ then $D(\frac{p}{q}) \in S_q$.

So why is $\frac{p}{q}$ eventually periodic? Let $x_0 = \frac{p}{q}$ and $x_i = D^i(x_0)$. By the above argument, each $x_i \in S_q$. There are q elements of the set S_q. So, consider the $q+1$ points

$$x_0, x_1, x_2, \ldots x_q.$$

By the pigeon hole principle, there must be indices i and j with $i < j$ and $x_i = x_j$. Then by definition, x_0 is eventually period-$(j-i)$!

Now suppose that $x_0 = \frac{p}{q}$ is not periodic. Then, there is a smallest $j > 0$ such that x_j is periodic. Let n denote the period of x_j. Therefore, we know that

$$D(x_{j-1}) = x_j \quad \text{and} \quad D(x_{n+j-1}) = x_{n+j} = x_j.$$

Now we will show that $x_{j-1} \neq x_{n+j-1}$. Notice that x_{j-1} is not periodic, because of our choice of j. On the other hand, x_{n+j-1} is period-n because

$$D^n(x_{n+j-1}) = D^{n-1}(x_{n+j}) = D^{n-1}(x_j) = x_{n+j-1}.$$

So, $x_{j-1} \neq x_{n+j-1}$. In summary, we have found distinct points $x_{j-1}, x_{n+j-1} \in S_q$ which are mapped to the same point $x_j \in S_q$ by D. Let $\frac{r}{q} = x_j$. Consider the possible values of $y \in [0,1)$ for which $D(y) = \frac{r}{q}$. If $y < \frac{1}{2}$, then $2y = \frac{r}{q}$ so we have the possible solution

$$y_1 = \frac{r}{2q}.$$

And if $y \geq \frac{1}{2}$ then $2y - 1 = \frac{r}{q}$ so we have the solution

$$y_2 = \frac{r+q}{2q}.$$

So $x_{j-1} = y_1$ and $x_{n+j-1} = y_2$ or vice versa. But this is a contradiction, because we know that both x_{j-1} and x_{j+n-1} lie in S_q, and can be written as fractions with q in the denominator. But this is impossible because the numerators of y_1 and y_2 can not both be even! $\square$

What we have really shown:
The proof above shows that the restriction map $D|_{S_q} : S_q \to S_q$ is injective (one-to-one). We have the following general proposition.

Proposition 1 *Let S be a finite set and $f : S \to S$. Then the following statements are equivalent:*

1. *f is injective (one-to-one).*
2. *f is surjective (onto).*
3. *All points in S are periodic.*

A map f satisfying these statements is often called a *permutation*.