Math-354: Bifurcations

We are interested in the following question: "For a *typical* family of functions $f_\mu : \mathbb{R} \to \mathbb{R}$, how can the local dynamics of f_μ change with μ?"

For us a family of functions is really a function $f : \mathbb{R}^2 \to \mathbb{R}$ of the form $f(x, \mu)$ with several continuous derivatives. We write $f_\mu(x) = f(x, \mu)$.

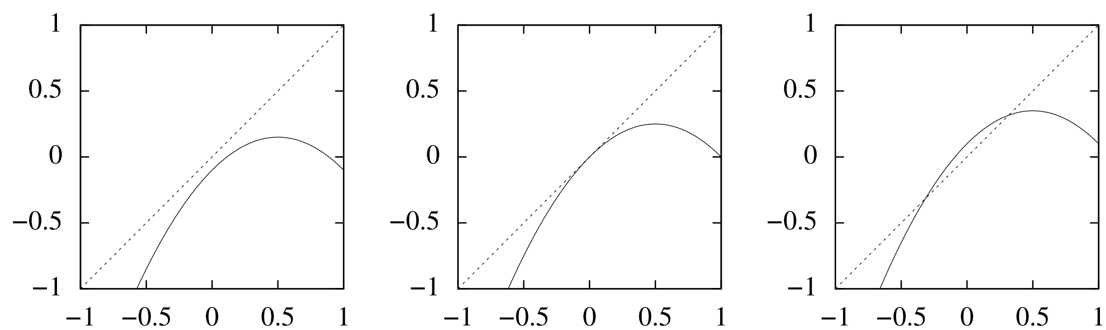
Tangential Bifurcation

A tangential bifurcation is what *typically* happens near a fixed point x_0 of f_{μ_0} with $f'_{\mu_0}(x_0) = 1$.

Example 9.5.1. Consider the family of functions

$$f_\mu(x) = \mu + x - x^2.$$

The fixed points are $p_+ = \sqrt{\mu}$ and $p_- = -\sqrt{\mu}$, which exist when $\mu \geq 0$. We have $f'_\mu(x) = 1 - 2x$, so p_- is always repelling when $\mu > 0$. On the other hand, p_+ is attracting when $0 < \mu < 1$ and repelling when $\mu > 1$. The graphs of f_μ for $\mu = -0.1, 0, 0.1$ are shown below.



The following is a weak version of the book's Theorem 9.5.2.

Theorem. *(Tangential Bifurcation)* Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function. Assume the following hold:

1. $f_{\mu_0}(x_0) = x_0$. (x_0 is fixed by f_{μ_0}.)
2. $f'_{\mu_0}(x_0) = 1$. (The graph of f'_{μ_0} is tangent to the diagonal near x_0.)

Also assume the following statements are true: (They will be true *typically*, but not always.)

(3) $f''_{\mu_0}(x_0) \neq 0$. (Near x_0, the graph looks like a parabola, and therefore the graph lies either above or below the diagonal.)

(4) $\frac{\partial f}{\partial \mu}(x_0, \mu_0) \neq 0$. (The path $\big(\mu, f_\mu(x_0)\big)$ passes through the diagonal as μ passes through the value μ_0.

Then there are constants $\epsilon > 0$ and a $r > 0$, such that for $|\mu - \mu_0| < \epsilon$ such that the following hold:

1. On one side of μ_0 (either $\mu < \mu_0$ or $\mu > \mu_0$), there are no fixed points of f_μ within distance r of x_0.

2. On the other side of μ_0, there are both attracting and repelling fixed points of f_μ within distance r of x_0. For these values of μ, the attracting and repelling fixed points will lie on opposite sides of x_0.

Theorem 9.5.2 in our book further describes properties of these fixed points.

Exercise. Suppose $f''_{\mu_0}(x_0) > 0$ and $\frac{\partial f}{\partial \mu}(x_0, \mu_0) > 0$. Do the fixed points exist for μ larger or smaller than μ_0? Is the attracting fixed point greater than or less than x_0?

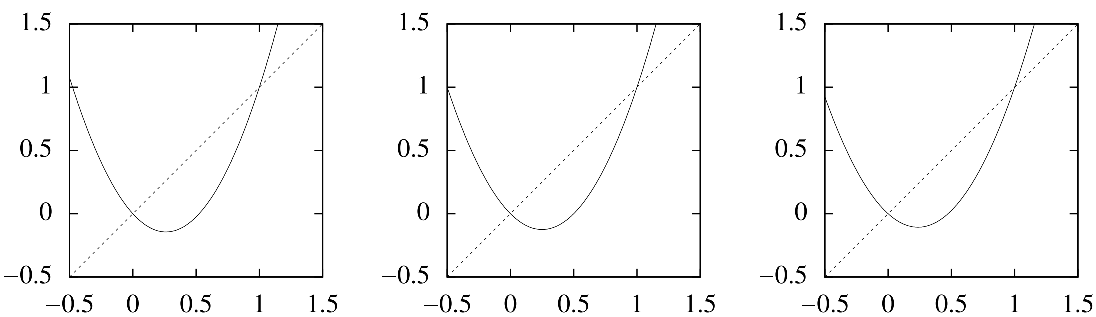
Period Doubling Bifurcation

The period doubling bifurcation explains the local dynamical behaviour of a *typical* family of functions f_μ with a fixed point x_0 of f_{μ_0} with $f'_{\mu_0}(x_0) = -1$.

Example. Consider the family

$$f_\mu(x) = (\mu - 2)x(1 - x) + x.$$

The fixed points are always 0 and 1. Let $x_0 = 0$. We have $f'_\mu(x) = (\mu - 2)(1 - 2x) + 1$. We have $f'_\mu(x_0) = \mu - 1$. So x_0 point is attracting when $0 < \mu < 2$, and repelling when $\mu < 0$ or $\mu > 0$. Set $\mu_0 = 0$ and note that $f'_{\mu_0}(x_0) = -1$. See the graphs of f_μ for the cases $\mu = -0.1$, $\mu = 0$, and $\mu = 0.1$ below.

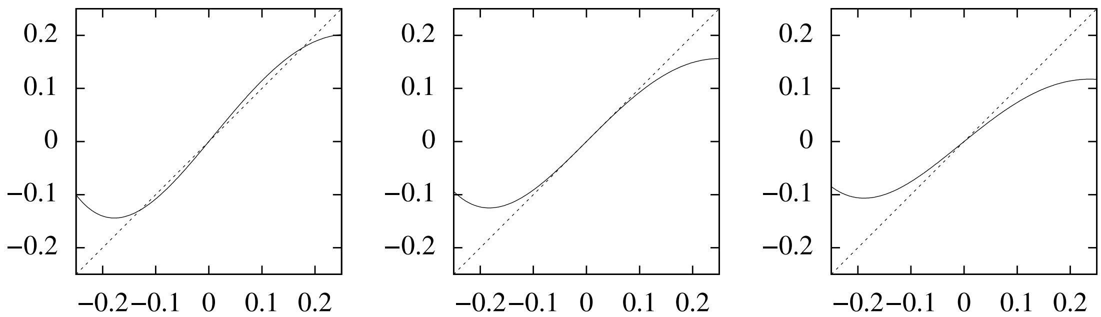


Unless you look very closely, you will detect little substantial differences between the graphs above. However, note that if $f'_{\mu_0}(x_0) = -1$ at the fixed point x_0, we have $(f^2_{\mu_0})'(x_0) = 1$. We should interpret this as meaning $f^2_{\mu_0}$ looks similar to the identity map near x_0. This motivates us to look at the square.

The square of this function can be given by the formula

$$f^2_\mu(x) = (\mu - 2)x(1 - x)\big(\mu - \mu(\mu - 2)x + (\mu - 2)^2 x^2\big) + x.$$

The graphs of f^2_μ near $x = 0$ are given below for the values $\mu = -0.1$, $\mu = 0$, and $\mu = 0.1$.



We see that in the case $\mu = -0.1$, there are two period-2 points. From our formula for $f_\mu^2(x)$, we see that if there are period two points, they must be roots of the equation $\mu - \mu(\mu-2)x + (\mu-2)^2x^2$. We compute these roots to be

$$p_\pm = \frac{\mu \pm \sqrt{\mu(\mu-4)}}{4-2\mu}.$$

These points are well defined when $\mu(\mu-4) > 0$. That is, when $0.44 \approx 2-\sqrt{6} < \mu < 0$. As these two points are the only period-2 points, they must form a period-2 orbit. We would like to determine the type of this orbit. We have

$$(f_\mu^2)'(p_\pm) = f_\mu'(p_+)f_\mu'(p_-) = 1 - \mu(\mu-4).$$

So, this orbit is attracting whenever $2-\sqrt{6} < \mu < 0$.

The bifurcation is connected to the fact that $f_{\mu_0}^2(x)$ has a third order tangency to the diagonal at the point x_0. That is,

$$f_{\mu_0}^2(x_0) = x_0, \quad (f_{\mu_0}^2)'(x_0) = 1, \quad (f_{\mu_0}^2)''(x_0) = 0, \quad \text{and} \quad (f_{\mu_0}^2)'''(x_0) \neq 0.$$

In general, we will be assuming facts that directly imply that $f_{\mu_0}^2(x_0) = x_0$ and $(f_{\mu_0}^2)'(x_0) = 1$. The facts about the other derivatives are more difficult. In fact with our assumptions we will also have $(f_{\mu_0}^2)''(x_0) = 0$. To see why this is true, we introduce the following notational tool.

For our purposes, big O-notation is a convienient way to collect terms of a function which are irrelevant to our undertanding of the function near a point of interest. (Terms which are very small near a point.)

Big O-notation. We write $f(x) = g(x) + O\big(h(x)\big)$ to mean that there is are constants $\epsilon > 0$ and $C > 0$ such that $|f(x) - g(x)| \leq C|h(x)|$ whenever $|h(x)| < \epsilon$. This is the same as writing $f(x) - g(x) = O\big(h(x)\big)$.

Example using Taylor's theorem. By Taylor's theorem, there is an $\epsilon > 0$ such that for any x with $|x| < \epsilon$, we have

$$e^x = 1 + x + \frac{x^2}{2} + R(x),$$

where $R(x) = \frac{\frac{d}{dx}f(\xi)}{6}x^3$ for some ξ between 0 and x. Since $\frac{d}{dx}f(\xi)$ is bounded for ξ in the interval $(-\epsilon, \epsilon)$, we can write

$$e^x = 1 + x + \frac{x^2}{2} + O(x^3).$$

Some properties of Big O. Heuristically, if $f(x) = O\big(h(x)\big)$, it means that $f(x)$ converges to zero faster or nearly as fast as $h(x)$.

1. If $f(x) = O\big(h(x)\big)$ and $g(x) = O\big(h(x)\big)$ then $f(x) + g(x) = O\big(h(x)\big)$.
2. If k is a constant and $f(x) = O\big(h(x)\big)$ then $kf(x) = O\big(h(x)\big)$.
3. If $0 < m \leq n$, then $x^n = O(x^m)$.
4. If $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is a polynomial and $a_0 = 0$, then $p(x) = O(x^k)$ where k is the smallest integer with $a_k \neq 0$. Also, $x^k = O\big(p(x)\big)$.

The square. Suppose g is a C^3 function of the form

$$g(x) = -x + ax^2 + bx^3 + O(x^4).$$

We have

$$\begin{aligned} g^2(x) &= -\big(-x + ax^2 + bx^3 + O(x^4)\big) + a\big(-x + ax^2 + bx^3 + O(x^4)\big)^2 + \\ &\quad b\big(-x + ax^2 + bx^3 + O(x^4)\big)^3 + O\Big(\big(-x + ax^2 + bx^3 + O(x^4)\big)^4\Big). \end{aligned}$$

Note that $O\Big(\big(-x + ax^2 + bx^3 + O(x^4)\big)^4\Big)$ is the same as $O(x^4)$. By grouping terms which are $O(x^4)$, we have

$$g^2(x) = (x - ax^2 - bx^3) + a(x^2 + 2ax^3) + b(-x^3) + O(x^4) = x + (-2a^2 - 2b)x^3 + O(x^4).$$

In particular, there is no quadratic term. This also explains how to compute the cubic term of g^2 as described in Theorem 9.5.4 of the book. The following is a weaker version of the theorem in the book.

Theorem. *(Period Doubling Bifurcation)* Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^3 function. Assume the following hold:

1. $f_{\mu_0}(x_0) = x_0$. (x_0 is fixed by f_{μ_0}.)
2. $f'_{\mu_0}(x_0) = -1$. (The graph of f'_{μ_0} has slope -1 at x_0.)

It follows from the implicit function theorem that there is a function x from a neighborhood of μ_0 to $\mathbb{R}$ such that $f_\mu\big(x(\mu)\big) = x(\mu)$ and $x(\mu_0) = x_0$. This function is a differentiable curve of fixed points. Also assume the following statements are true: (They will be true *typically*, but not always.)

(3) The derivative of $f'_\mu\big(x(\mu)\big)$ with respect to μ is nonzero at (x_0, μ_0). (The slope of the tangent line of the graph of f_μ is varying nearly linearly with μ. This also implies that the slope of the tangent line of f^2_μ at the fixed point of f_μ is varying nearly linearly with μ.)

(4) The third derivative $(f^2_{\mu_0})'''(x_0)$ is nonzero. (This says that the graph of f_μ looks very much like the graph of a cubic polynomial in a neighborhood of x_0 for values of μ near μ_0.)

Then there are constants $\epsilon > 0$ and a $r > 0$, such that for $|\mu - \mu_0| < \epsilon$ such that the following hold:

1. The type of the fixed point $x(\mu)$ switches between attracting and repelling as μ passes through μ_0.
2. On one side of μ_0 (either $\mu < \mu_0$ or $\mu > \mu_0$), there is an additional period-2 orbit within r of x_0. In this case, when $x(\mu)$ is repelling, the period-2 orbit will be attracting. Conversely, when $x(\mu)$ is attracting, the period-2 orbit will be repelling.
3. These are the only fixed and period-2 points within r of x_0.

The theorem in the book describes properties of these fixed and periodic points in much greater detail.

Exercise. Assume $\alpha = \frac{d}{d\mu} f'_\mu\big(x(\mu)\big)|_{\mu=\mu_0} > 0$ and $\beta = (f^2_{\mu_0})'''(x_0) > 0$. On which side of μ_0 is $x(\mu)$ attracting? Which side of μ_0 has period-2 points?