Math-354: Some Homework Problems from §10.2

10.2.1(a). Let $S_1 = \{\frac{p}{2^n} \in [0,1] \ : \ \text{both } p \text{ and } n \text{ are positive integers}\}$. We will show S_1 is dense in $[0,1]$ by using the definition of dense. Let $z \in [0,1]$ and $\epsilon > 0$. We need to show that $S_1 \cap (z-\epsilon, z+\epsilon) \neq \emptyset$.

We will write the binary expansion of z. We can find numbers $z_i \in \{0,1\}$ for integers $i \geq 1$ so that

$$z = \sum_{i=1}^{\infty} \frac{z_i}{2^i}.$$

Since $\lim_{n\to+\infty} \frac{1}{2^n} = 0$, we can find an n for which $\frac{1}{2^n} < \epsilon$. Define

$$a = \sum_{i=1}^{n} \frac{z_i}{2^i}.$$

Then $a \in S_1$. Also $z - a = \sum_{i=n+1}^{\infty} \frac{z_i}{2^i} > 0$. Because we chose n so that $\frac{1}{2^n} < \epsilon$, we have

$$z - a = \sum_{i=n+1}^{\infty} \frac{z_i}{2^i} \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n} < \epsilon.$$

Therefore $a \in (z-\epsilon, z+\epsilon)$. We conclude that $S_1 \cap (z-\epsilon, z+\epsilon) \neq \emptyset$ for all z and ϵ.

10.2.1(b). Let $S_2 = [0,1] \setminus S_1$. We will prove that S_2 is dense in $[0,1]$. Let $z \in [0,1]$ and $\epsilon > 0$. Choose any pair of numbers $a < b$ such that $a, b \in (z-\epsilon, z+\epsilon) \cap [0,1]$. If $a \notin S_1$, then $a \in (z-\epsilon, z+\epsilon) \cap S_2$. So if $a \notin S_1$, then $S_2 \cap (z-\epsilon, z+\epsilon) \neq \emptyset$ as desired. So, we may assume that $a \in S_1$. Similarly, we may assume $b \in S_1$. In particular, both a and b are rational. Let $c = a + \frac{\sqrt{2}}{2}(b-a)$. Then c is irrational because $a, b \in \mathbb{Q}$ and $b - a \neq 0$. Therefore $c \in S_2$. Also since $0 < \frac{\sqrt{2}}{2} < 1$, we know $a < c < b$. Therefore, $c \in S_2 \cap (z-\epsilon, z+\epsilon)$, and this intersection is nonempty.

10.2.1(c). Let $S_3 = \left\{x = \sum_{j=1}^{\infty} \frac{d_j}{10^j} \ : \ d_j \in \{0,2,4,6,8\}\right\}$. We claim S_3 is not dense. It is enough to show that $S_3 \cap (\frac{1}{10}, \frac{2}{10}) = \emptyset$, since dense sets in $[0,1]$ must intersect every open interval in $[0,1]$.

Suppose $x = \displaystyle\sum_{j=1}^{\infty} \frac{d_j}{10^j} \in S_3 \cap (\frac{1}{10}, \frac{2}{10})$. Since $x < \frac{2}{10}$, we know that $d_1 = 0$. Therefore,

$$x = \sum_{j=2}^{\infty} \frac{d_j}{10^j} \leq \sum_{j=2}^{\infty} \frac{8}{10^j} < \sum_{j=2}^{\infty} \frac{9}{10^j} = \frac{1}{10}.$$

But the statement that $x < \frac{1}{10}$ is a contradiction, since we assumed $x \in (\frac{1}{10}, \frac{2}{10})$.

10.2.4. Find an irrational number that does not have a dense orbit under the doubling map, D. (Note that the problem did not ask for a proof, but I have included one.)

Recall that rationals are all periodic or eventually periodic under the doubling map. (See example 9.1.6 in the book.) So, it is enough to find a number x whose orbit is not dense, and which is not eventually periodic. For integers $j \geq 1$ define

$$x_j = \begin{cases} 1 & \text{if } \sqrt{j} \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases}$$

Then define

$$x = \sum_{j=1}^{\infty} \frac{x_j}{2^j} = \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^9} + \dots.$$

First we will show that the orbit of x is not dense. Note that if $x_j = 1$ then $x_{j+1} = 0$. (In other words, the binary expansion of x has no adjacent ones.) For any integer k, we have

$$D^k(x) = \sum_{j=1}^{\infty} \frac{x_{j+k}}{2^j}.$$

Therefore, the first two terms in the binary expansion of $D^k(x)$ cannot be $\frac{1}{2} + \frac{1}{4}$. It follows that $D^k(x) \leq \frac{3}{4}$ for all k. Thus, the orbit of x does not intersect the open interval $(\frac{3}{4}, 1)$. So this orbit is not dense.

We now claim that x is not eventually periodic under the doubling map. To see this we will first prove that for all $\epsilon > 0$ there is a k such that $D^k(x) < \epsilon$. We can find an integer N such that $\frac{1}{2^{2N}} < \epsilon$. Let $k = N^2$. Then

$$D^{N^2}(x) = \sum_{i=1}^{\infty} \frac{x_{i+N^2}}{2^i}.$$

By definition of x_j, we know that $x_j = 0$ for all values of j satisfying $N^2 < j < (N+1)^2$. Therefore, for $1 \leq i < (N+1)^2 - N^2 = 2N+1$ we have $x_{i+N^2} = 0$. Thus,

$$D^{N^2}(x) = \sum_{i=2N+1}^{\infty} \frac{x_{i+N^2}}{2^i} \leq \sum_{i=2N+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^{2N}} < \epsilon.$$

Now suppose x is periodic or eventually periodic. Then the orbit $\mathcal{O}_f^+(x)$ is a finite set. Let y be the smallest element of this set. Note that $y > 0$, because 0 is not in the orbit of x. Let $\epsilon = y$. Then the previous paragraph produces a k for which $D^k(x) < \epsilon = y$. But, this contradicts the assumption that y is the smallest element in the orbit.