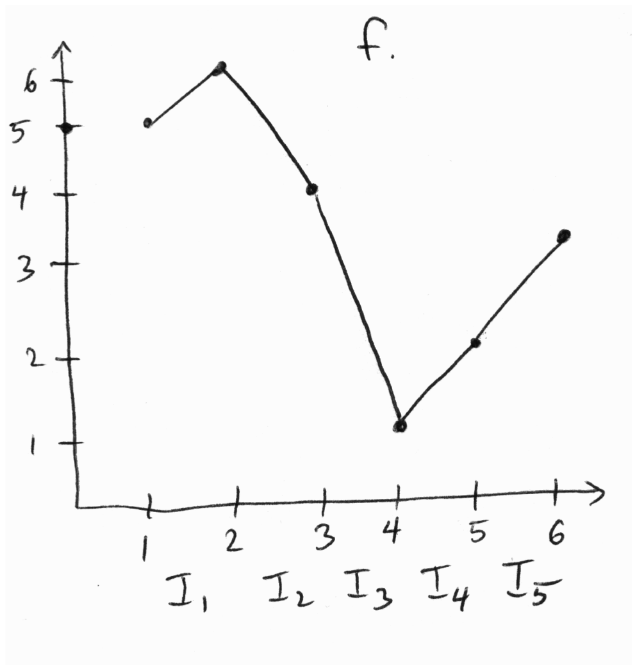
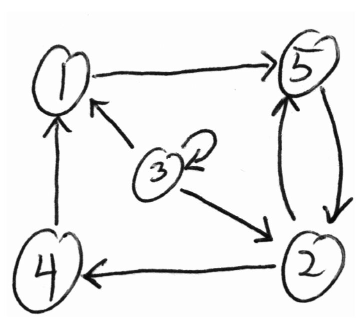
Math-354: Homework Problem 10.1.2

Remark. The answer provided here goes beyond the question asked in the book. I hope this extra information will help you better understand these types of problems.
10.2.1(a). The graph of f is sketched below.



10.2.1(b). Let $I_1 = [1, 2]$, $I_2 = [2, 3]$, $I_3 = [3, 4]$, $I_4 = [4, 5]$, and $I_5 = [5, 6]$. The transition graph is shown below.



10.2.1(c).

- **What periods can occur?**
  The set of end points $\{1, 2, 3, 4, 5, 6\}$ forms an orbit of period-6. So Sharkovskii's theorem implies that f has a fixed point and points of all even periods.

- **Claim.** Let $n \geq 3$ be odd. Then f has no point of period n.
  Proof. Suppose p_0 is a point of period-n. Let $p_i = f^i(p_0)$ for $i = 0, \dots, n$. Then no $p_i \in \{1, 2, 3, 4, 5, 6\}$ because p_i is not period-6. Thus, each p_i lies in the interior of a unique interval I_{s_i}. In particular, we have the periodic allowable string $s_0 s_1 s_2 \dots s_n$.

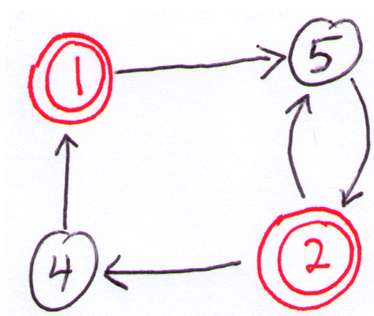
(This is part of the itinerary of p_0, and $p_0 = p_n$ so $s_0 = s_n$.) Note that if some $s_i = 3$, then all $s_i = 3$, because the only periodic string which contains 3 is the path of all 3s. Thus, we have to rule out two cases.

Suppose that each $s_i = 3$. In particular, each $f^i(p_0) \in I_3$. Note that there is a fixed point $\frac{13}{4} \in I_3$. For all $a \in I_3$, we have $f(a) = 13 - 3a$. Then for any $a \in I_3$ we have $|f(a) - f(\frac{13}{4})| = 3|a - \frac{13}{4}|$. Since p_0 is period-n and $\frac{13}{4}$ is fixed, we know $p_0 \neq \frac{13}{4}$. Since each $f^i(p_0) \in I_3$, we have

$$|f^i(p_0) - \frac{13}{4}| = 3^k|p_0 - \frac{13}{4}|,$$

which is arbitrarily large. This is a contradiction to the assumption that all $f^i(p_0) \in I_3$.

Now suppose that no $s_i = 3$. Then the string $s_0s_1 \ldots s_n$ is a periodic allowable string consisting only of the symbols in $\{1, 2, 4, 5\}$. Color the vertices 1 and 2 red and the vertices 4 and 5 black. Then every edge between vertices in $\{1, 2, 4, 5\}$ joins a black vertex to a red vertex or vice versa. (See the colored graph below.) In particular, since $s_0s_1 \ldots s_n$ is allowable, the colors of vertices must alternate. In particular, since n is odd, the color of s_0 is the opposite of the color of s_n. This is a contradiction, since for $s_0s_1 \ldots s_n$ to be periodic, we must have $s_0 = s_n$.



- **Itineraries of periodic orbits.**
  Consider the period-1 string "33". Theorem 10.1.8 guarantees that there is a point $p \in I_3$ for which $f(p) = p$.

  Consider the period-2 string "252". Again, Theorem 10.1.8 guarantees that there is a $p \in I_2$ for which $f^2(p) = p$ and $f(p) \in I_5$. Since I_2 and I_5 are disjoint, p cannot be a fixed point.

  The $\{1, 2, 3, 4, 5, 6\}$ is a period-6 orbit.

  Let k be any positive integer other than 1 or 3, and let $n = 2k$. Consider the period-n string "$(52)^{k-1}415$"$=s_0s_1 \ldots s_n$. Since the string contains only one "1", the string is irreducible. Theorem 10.1.8 guarantees that there is a $p \in I_5$ such that $f^n(p) = p$, and $f^i(p) \in I_{s_i}$ for all $i = 0, \ldots, n$. Each endpoint of each interval is period-6. Therefore, if n is not a multiple of 6, then p cannot be an endpoint of an interval. Thus, by Theorem 10.1.8(d), we know p is period-n. Suppose n is a multiple of 6, but $n \geq 12$ and so $k \geq 6$. We claim p is not an endpoint. Suppose p is an endpoint. Then because $p \in I_5$, $p = 5$ or $p = 6$. Suppose $p = 5$. Then $f(p) = 2 \in I_2$, $f^2(p) = 6 \in I_5$, $f^3(p) = 3 \in I_2$, and $f^4(p) = 4 \notin I_5$. So, $p = 5$ cannot be given the itinerary "$(52)^{k-1}415$". Similarly, suppose $p = 6$. Then, $f(p) = 3 \in I_2$ and $f^2(p) = 4 \notin I_5$. So $p = 6$ cannote be given the itinerary "$(52)^{k-1}415$". We conclude that p is not an endpoint. Thus, by Theorem 10.1.8(d), we know p is period-n.

- **What are we worried about?**
  It might seem that we are concerned about endpoints for no reason. Consider that the itinerary of the point $q = 1$ can be interpreted to be any string formed by taking one row from each column in the following

$$1 \begin{matrix} 4 \\ 5 \end{matrix} \begin{matrix} 1 \\ 2 \end{matrix} 5 \begin{matrix} 2 \\ 3 \end{matrix} \begin{matrix} 3 \\ 4 \end{matrix} 1$$

  For example the period-12 string "1525241415241" is a coding for the orbit of $q = 1$. Note that this string looks irreducible of period-12, but is in fact coding a point of period 6. (In this particular example, this string is not allowable. But, more elaborate functions can be found which have an irreducible allowable periodic string, which is a coding for an endpoint of smaller period.)