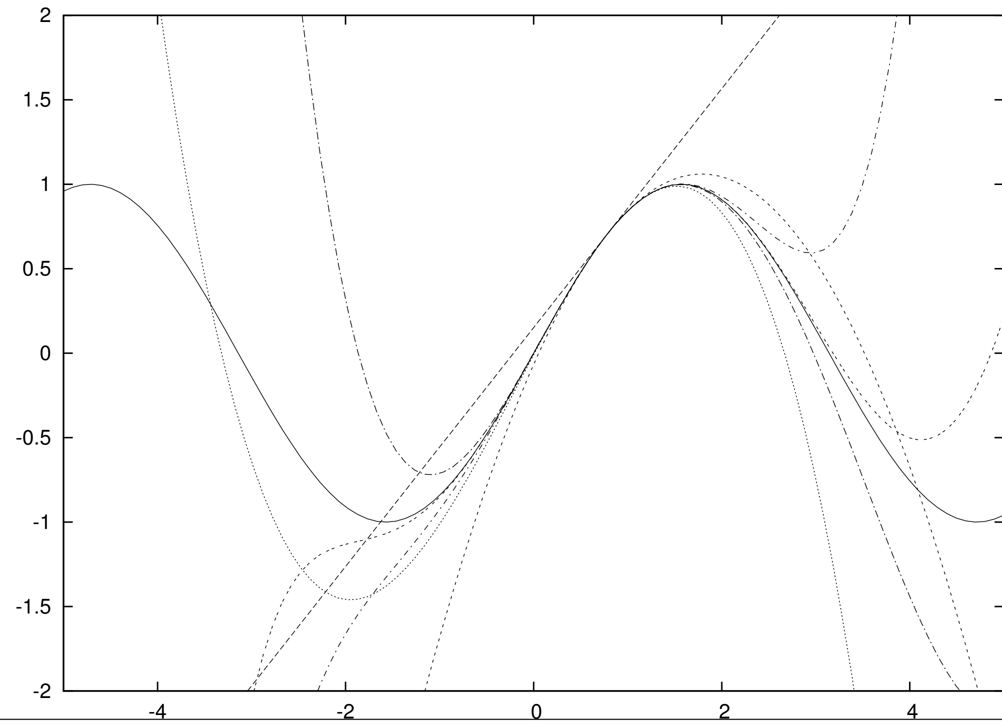
The first few Taylor approximations to $sin(x)$ at $x = \frac{\pi}{4}$



Taylor's theorem in one variable

Theorem: Let $X \subset \mathbb{R}$ be open, and $f : X \to \mathbb{R}$ be differentiable of order k. Given $a \in X$, let

$$p_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Then, the remainder term $R_k(x,a) = f(x) - p_k(x)$ satisfies

$$\lim_{x \to a} \frac{R_k(x,a)}{(x-a)^k} = 0.$$

First order Taylor's formula in several variables

Theorem: Let $X \subset \mathbb{R}^n$ be open, and $f : X \to \mathbb{R}$. Suppose that f is differentiable at $\mathbf{a} \in X$. Let

$$p_1(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}).$$

Then the remainder $R_1(\mathbf{x}, \mathbf{a}) = f(\mathbf{x}) - p_1(\mathbf{x})$ satisfies

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{R_1(\mathbf{x}, \mathbf{a})}{\|\mathbf{x} - \mathbf{a}\|} = 0.$$

Second order: Approximation by a quadric surface

Let $a, b, c, d, e, f \in \mathbb{R}$ be constants and let

$$F(x, y) = a + bx + cy + dx^2 + exy + fy^2.$$

The graph of F is a *quadric surface*.

We have $F(0,0) = a$, $F_x(0,0) = b$, $F_y(0,0) = c$, $F_{xx}(0,0) = 2d$, $F_{xy}(0,0) = e$, $F_{yx}(0,0) = e$, and $F_{yy}(0,0) = 2f$. So, if

$$\begin{aligned} p_2(x, y) \quad = \quad & F(0,0) + F_x(0,0)x + F_y(0,0)y + \\ & \tfrac{F_{xx}(0,0)}{2}x^2 + \tfrac{F_{xy}(0,0)}{2}xy + \tfrac{F_{yx}(0,0)}{2}yx + \tfrac{F_{yy}(0,0)}{2}y^2, \end{aligned}$$

then $p_2 = F$.

In fact this $p_2(x, y)$ works for general functions to give the closest approximating quadric surface at $(0,0)$ to a general C^2 function $F(x, y)$.