Math290-2, Section 61:
An example quadratic form problem

Problem: Consider the quadratic form $Q(\mathbf{x}) = 73x_1^2 - 72x_1x_2 + 52x_2^2$.

1. Find the matrix A for the quadratic form.
2. Find a change of coordinates, $\mathbf{x} = P\mathbf{y}$, that transforms Q into a quadratic form with no cross term.
3. Graph the level set $Q(\mathbf{x}) = 100$.

Solution:

1. The matrix $A = \begin{bmatrix} 73 & -36 \\ -36 & 52 \end{bmatrix}$ suffices.

2. To transform Q in this way it is enough to orthogonally diagonalize A. We will find an orthogonal matrix P (where $P^{-1} = P^T$) and a diagonal matrix D such that

$$A = PDP^T = PDP^{-1}.$$

Then the substitution $\mathbf{x} = P\mathbf{y}$ transforms Q in the correct way. This is because

$$\begin{aligned} Q(\mathbf{x}) &= Q(P\mathbf{y}) \\ &= (P\mathbf{y})^T A(P\mathbf{y}) && \text{by definition of } Q \\ &= (\mathbf{y}^T P^T)A(P\mathbf{y}) && \text{by the property } (AB)^T = B^T A^T \\ &= (\mathbf{y}^T P^T)PDP^T(P\mathbf{y}) && \text{by substitution } A = PDP^T \\ &= \mathbf{y}^T(P^{-1}P)D(P^{-1}P)\mathbf{y} && \text{by substitution } P^T = P^{-1} \text{ and regrouping} \\ &= \mathbf{y}^T D\mathbf{y} \end{aligned}$$

So, let's orthogonally diagonalize A. We do the following steps.

(a) Compute the eigenvalues. We will factor the characteristic polynomial.

$$\det(A - \lambda I) = (73 - \lambda)(52 - \lambda) - (-36)^2 = 2500 - 125 + \lambda^2 = (\lambda - 25)(\lambda - 100).$$

So the eigenvalues must be $\lambda_1 = 25$ and $\lambda_2 = 100$.

(b) Compute the corresponding eigenvectors.

$\boldsymbol{\lambda_1}$: The eigenspace is corresponding to λ_1 is $Nul(A - \lambda_1 I)$. We row reduce $A - \lambda_1 I$.

$$A - 25I = \begin{bmatrix} 48 & -36 \\ -36 & 27 \end{bmatrix} \sim \begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix} \sim \begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix}.$$

So, $Nul(A - \lambda_1 I) = \text{Span}\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}\}$.

λ_2 : The eigenspace is corresponding to $\lambda_2 = 100$ is $Nul(A - \lambda_2 I)$. We row reduce $A - \lambda_1 2$.

$$A - 100I = \begin{bmatrix} -27 & -36 \\ -36 & -48 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}.$$

So, $Nul(A - \lambda_1 I) = \text{Span}\{\begin{bmatrix} -4 \\ 3 \end{bmatrix}\}$.

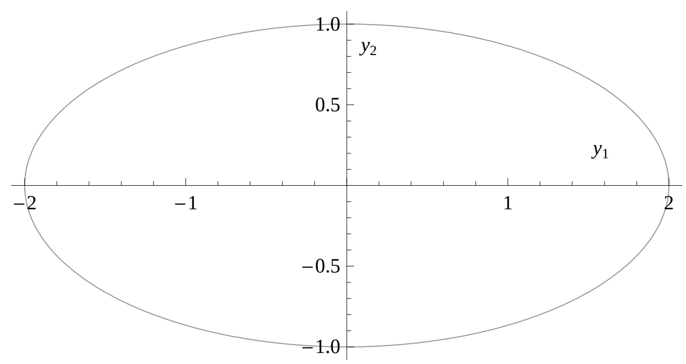
(c) Find an orthonormal set of eigenvectors. We choose $\mathbf{u}_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ from the first eigenspace and $\mathbf{u}_2 = \begin{bmatrix} \frac{-4}{5} \\ \frac{3}{5} \end{bmatrix}$ from the second.

(d) Form the orthogonal diagonalization. Let

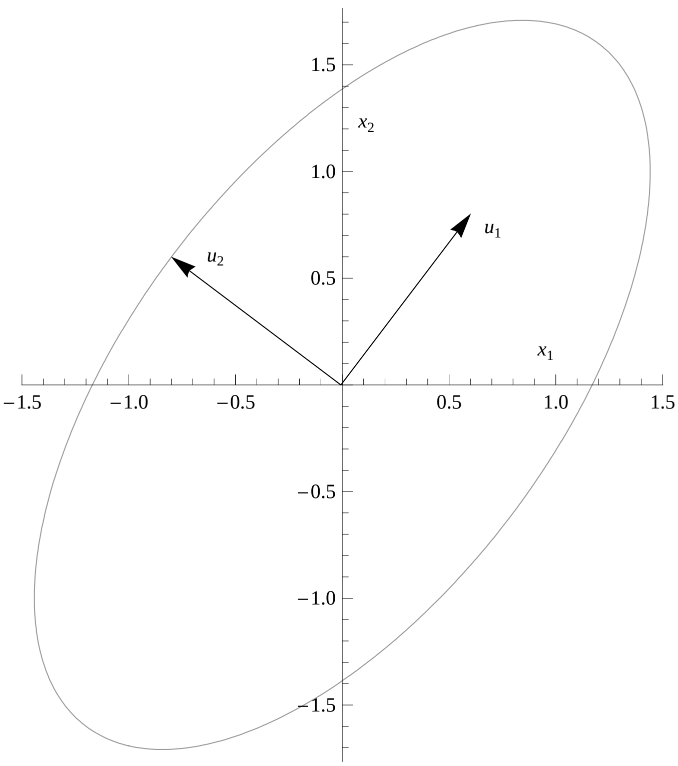
$$P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 100 \end{bmatrix}.$$

(e) The change of variables we want is $\mathbf{x} = P\mathbf{y}$, with P as above.

3. First we graph in $\mathbf{y}$-coordinates, then in $\mathbf{x}$-coordinates. In $\mathbf{y}$-coordinates, Q has the simple form $25y_1^2 + 100y_2^2$. We draw the level set $Q = 100$ in $\mathbf{y}$-coordinates. The graph is an ellipse which passes through the points $(2,0)$, $(0,1)$, $(-2,0)$, $(0,-1)$.



We can draw the graph in $\mathbf{x}$-coordinates using the equation $\mathbf{x} = P\mathbf{y}$. The graph is an ellipse which passes through the points $P(2,0) = 2\mathbf{u}_1$, $P(0,1) = \mathbf{u}_2$, $P(-2,0) = -2\mathbf{u}_1$, $P(0,-1) = -\mathbf{u}_2$.



1.5
1.0
0.5
−0.5
−1.0
−1.5
−1.5
−1.0
−0.5
0.5
1.0
1.5
x_1
x_2
u_1
u_2