The final will cover the following sections.

- Lay §6.1-6.3 and §7.1-7.2
- Colley §1.1-1.7, §2.1-2.6, §4.1-4.2

Practice Questions:

1. Complete each definition.
   (a) "A matrix A is **orthogonally diagonalizable** if ..."
   (b) "Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$. Let $c \in \mathbb{R}$. The **level set of f at height** c is ..."
   (c) A matrix A is **symmetric** if ...
   (d) "Suppose $X \subset \mathbb{R}^n$ is open and $f : X \to \mathbb{R}$. The function f has a **local maximum** at $\mathbf{a} \in X$ if ..."
   (e) "Let X be an open set in $\mathbb{R}^n$ and let $\mathbf{a} \in X$. A function $f : X \to \mathbb{R}$ has a **local minimum** at $\mathbf{a}$ if ..."

2. True or False.
   (a) There is a differentiable function f such that $D_{\mathbf{u}}f(0,0)$ is defined and positive for all unit vectors $\mathbf{u} \in \mathbb{R}^2$.
   (b) If $\mathbf{a} \neq \mathbf{0} \in \mathbb{R}^3$ then the set $\{\mathbf{a}\times\mathbf{i}, \mathbf{a}\times\mathbf{j}, \mathbf{a}\times\mathbf{k}\}$ is linearly independent.
   (c) The surface S defined by the equation $e^{z(x^2+y^2-1)} + z = x^2$ contains the curve parameterized by $\mathbf{r}(\theta) = (\cos\theta, \sin\theta, -1+\cos^2\theta)$.
   (d) If $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = 0$ then $\mathbf{c} \in \text{Span}\,\{\mathbf{a}, \mathbf{b}\}$.
   (e) The quadratic form with standard matrix $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is positive definite.
   (f) If f is a scalar-valued function with domain $\mathbb{R}^2$ so that both $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ exist, then f is differentiable at $(0,0)$.
   (g) Near the point $(2,2)$, the function $f(x,y) = x^2 + xy^2 + 2y$ is most sensitive to changes in x.
   (h) If f is a scalar-valued function with domain $\mathbb{R}^2$, then $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.
   (i) Every $n\times n$ matrix A with the property that $A^2 = A$ is the matrix of an orthogonal projection onto a subspace of $\mathbb{R}^n$.
   (j) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable at $(0,0,0)$. The tangent plane to the level set of f through $(0,0,0)$ is the null space of $Df(0,0,0)$.
   (k) There is a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ so that $\dfrac{\partial f}{\partial x}(x,y) = xy$ and $\dfrac{\partial f}{\partial y}(x,y) = xy$.
   (l) Every continuous function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ has a global minimum.

3. Suppose $g : \mathbb{R}^2 \to \mathbb{R}$ is of class C^1 and that $Dg(0,1) \neq \mathbf{0}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = g(xy, x^2)$. Prove that f does not have a local extremum at $(1,0)$.

4. Consider the function $\mathbf{r}(t) = (\cos t, \sin t, t)$ and the formula for the spherical coordinate $\rho(x,y,z) = \sqrt{x^2+y^2+z^2}$.
   (a) Compute the matrices of parial derivatives, $D\mathbf{r}$ and $D\rho$.
   (b) Use the chain rule to compute $\frac{d}{dt}[\rho \circ \mathbf{r}]$ as a function of t.
   (c) Consider the spherical coordinate φ as a function of x, y and z. Use substitution and the fact that $\tan\varphi = \frac{\sqrt{x^2+y^2}}{z}$ to compute $\frac{d}{dt}[\varphi \circ \mathbf{r}]$ as a function of t. (*Hint:* Recall that $\frac{d}{d\theta}\tan\theta = \sec^2\theta$.)

5. M is a real symmetric matrix with the eigenvalues 1 and 2. The eigenspace corresponding to the eigenvalue 2 is one dimensional and contains the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$. The eigenspace corresponding to the eigenvalue 1 is two dimensional and contains the vector $\mathbf{w} = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$.
   (a) Find a unit vector orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.
   (b) Orthogonally diagonalize the matrix M. That is, find an orthogonal matrix P and a diagonal matrix D such that $M = PDP^{-1}$.
   (c) Consider the line L with the parametric equation $\mathbf{r}(t) = t\mathbf{v} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find a parametric equation for the surface consisting of the set of points of distance 14 from L.
   (d) Name the surface described in part (c).

6. Let $f(x,y) = \sqrt{|xy|}$.
   (a) Find the unit vector $\mathbf{u}$ which maximizes $D_{\mathbf{u}}(3,12)$.
   (b) Let $\mathbf{u} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Evaluate the directional derivative $D_{\mathbf{u}}(3,12)$.
   (c) Compute the partial derivatives $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.
   (d) Use the definition of differentiable to prove that f is not differentiable at the origin.

7. Consider the function $g(x,y) = 3x - x^3 - 3xy^2$.
   (a) Show that $(0,-1)$ is a critical point of g.
   (b) Compute the second order Taylor polynomial at $(0,-1)$.
   (c) Classify the critical point $(0,-1)$.

8. Consider the function $\mathbf{r}(t,\theta) = (2\sqrt{t^2+1}\cos\theta, \sqrt{t^2+1}\sin\theta, t)$.

(a) The range of this function is a surface S. Find a function $f(x, y, z)$ such that the range of $\mathbf{r}$ is the same as the level set $f(x, y, z) = 0$.

(b) Sketch the intersection of this surface with the plane $z = 2\sqrt{2}$. (You may draw this in the plane.)

(c) Sketch the surface S. What is the name of the surface?

9. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$, or show that the limit does not exist.

10. The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = 10e^{-(x^2+3y^2)}$$

describes the temperature in degrees Celsius at a point (x, y) on a patch of snow where x and y are measured in meters. Wally the Walrus is wallowing in the snow at the point $(2, -1)$.

(a) In which direction should Wally waddle to warm up most quickly?

(b) If Wally waddles through the snow so that the path $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\mathbf{x}(t) = (2e^t, 3t - 1)$$

describes his position at time t measured in minutes, then what is the rate of change in temperature that Wally feels as he waddles through the point $(2e^4, 11)$? Provide units for your answer.

11. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = e^{xy}$.

(a) Find $Df(x, y)$.

(b) Find the critical points of f.

(c) Find the Hessian matrix $Hf(0, 0)$.

(d) Find the second-order Taylor approximation to f at $(0, 0)$.

(e) Orthogonally diagonalize $Hf(0, 0)$.

(f) Classify $Hf(0, 0)$ as positive definite, negative definite or indefinite.

(g) Find the principal axes of $Hf(0, 0)$.

(h) Define $Q : \mathbb{R}^2 \to \mathbb{R}$ by

$$Q(\mathbf{x}) = \mathbf{x}^T Hf(0, 0)\mathbf{x}.$$

Describe and sketch the level curves of Q.

(i) Describe and sketch the graph of Q.

(j) Classify all the critical points of f.