Math290-2, Section 61:
Differentiation and some "pathological" examples

1. Is the function $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ differentiable at $(0,0)$?

   It does have partial derivatives at $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

   and $\frac{\partial f}{\partial y}(0,0) = 0$. But is not differentiable, because it is not continuous at $(0,0)$. The limit

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

   does not exist.

2. Is the function $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ differentiable at $(0,0)$?

   This function is continuous, so we cannot use the argument above. But we can compute the partial derivatives. By working it out just like above we get

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

   So, our linear approximation would be $h(x,y) = 0$. But, is this a good approximation? We take the limit:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - h(x,y)}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{x^2y}{(x^2+y^2)^{\frac{3}{2}}} = \lim_{r\to 0} \cos^2\theta \sin\theta.$$

   This limit does not exists, so h is not a good approximation for f. Thus f is not differentiable at $(0,0)$.

3. Is the function $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ differentiable at $(0,0)$?

   This function is continuous and its partial derivatives are

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

   Again the linear approximation would be $h(x,y) = 0$. Is it good? We take the limit

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - h(x,y)}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}} = \lim_{r\to 0} r\cos^2\theta \sin\theta = 0.$$

   So it is a good approximation. This f is differentiable!

Actually we could see this in another way. The partial derivatives of f are

$$f_x(x,y) = \begin{cases} \frac{2xy^4}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \quad \text{and} \quad f_y(x,y) = \begin{cases} \frac{2x^4y}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

So, its partial derivatives are continuous. (We have to check using a limit.) So by the theorem in the text, f is differentiable.

4. Is the function $f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ differentiable at $x = 0$?

   It is continuous at 0. And the derivative of f at 0 is defined.

$$f'(x) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin(1/h)}{h} = \lim_{h \to 0} h \sin(1/h) = 0,$$

   by the squeeze theorem. The squeeze theorem applies, since $-|h| \leq h \sin \frac{1}{h} \leq |h|$, and $\lim_{h \to 0} \pm |h| = 0$.

   The complete formula for the derivative of f is

$$f'(x) = \begin{cases} 2x \sin(1/x) - \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   This is not a continous function. In fact the limit $\lim_{x \to 0} f'(x)$ does not exist. However, we check the definition of differentiability at $x = 0$. Our candidate linear approximation for f at zero is $h(x) = 0$. We check

$$\lim_{x \to 0} \frac{f(x) - h(x)}{|x|} = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{|x|} = \lim_{x \to 0} |x| \sin(1/x) = 0,$$

   by the squeeze theorem again. Here $-|x| \leq |x| \sin(1/x) \leq |x|$. So, f is differentiable, despite its partial derivatives being discontinuous.

5. The previous example was a function of one variable. If you want a similar example for a function of two variables consider

$$f(x,y) = \begin{cases} (x^2 + y^2) \sin(\frac{1}{\sqrt{x^2+y^2}}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   This has discontinuous partial derivatives, but it is differentiable.