Topological definitions

Recall these definitions from Colley's §2.2 (limits).

Definition: The **open ball** of radius r centered at $\mathbf{a} \in \mathbb{R}^n$ is the set

$$\{\mathbf{x} \in \mathbb{R}^n \text{ such that } \|\mathbf{x} - \mathbf{a}\| < r\}.$$

Definition: A set $X \subset \mathbb{R}^n$ is **open** if, for each point $\mathbf{x} \in X$, there is a open ball centered at $\mathbf{x}$ contained in X.

Definition: A **neighborhood** of $\mathbf{x} \in X$ is an open set containing x and contained in X.

Definitions of local maxima and minima

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$.

Definition: f has a **local minimum** at a point $\mathbf{a} \in X$ if there is a neighborhood U of $\mathbf{a}$ such that $f(\mathbf{x}) \geq f(\mathbf{a})$ for all $\mathbf{x} \in U$.

Similarly, f has a **local maximum** at a point $\mathbf{a} \in X$ if there is a neighborhood U of $\mathbf{a}$ such that $f(\mathbf{x}) \leq f(\mathbf{a})$ for all $\mathbf{x} \in U$.

Informally, a local minimum is a location $\mathbf{a}$ where all nearby points $\mathbf{x}$ satisfy $f(\mathbf{x}) \geq g(\mathbf{a})$.

Note: Because equality is allowed, we consider every point of the plane a local maximum and local minimum of the function $f(x, y) = 1$.

Definition: $\mathbf{a}$ is a **local extremum** of f if it is either a local maximum or a local minimum.

Recall the situation in one variable

Let $X \subset \mathbb{R}$ be open and $f : X \to \mathbb{R}$.

Theorem

Assuming f is differentiable, local extrema only occur at critical points. (A **critical point** *is an $x \in X$ where $f'(x) = 0$.)*

Theorem (Second derivative test)

Assume f is twice differentiable. And assume x is a critical point of f.

- *If $f''(x) > 0$ then x is a local minimum.*
- *If $f''(x) < 0$ then x is a local maximum.*
- *If $f''(x) = 0$ then the test is inconclusive.*

A quick one variable example

Let $f(x) = x^5 - x^3$. Then $f'(x) = 5x^4 - 3x^2$. So, f has the critical points 0 and $\pm\sqrt{\frac{3}{5}}$. We compute $f''(x) = 20x^3 - 6x$.

$f''(\sqrt{\frac{3}{5}}) = 6\sqrt{\frac{3}{5}} > 0$, so $\sqrt{\frac{3}{5}}$ is a local min.

$f''(-\sqrt{\frac{3}{5}}) = -6\sqrt{\frac{3}{5}} < 0$, so $-\sqrt{\frac{3}{5}}$ is a local max.

$f''(0) = 0$, so the test is inconclusive. In fact 0 is a saddle point. (It looks like $-x^3$ near 0.)

Recalling quadratic forms

Let H be a symmetric $n \times n$ matrix. The **quadratic form** associated to H is the map $Q(\mathbf{h}) = \mathbf{h}^T H\mathbf{h}$.

Q and H are **positive definite** if $Q(\mathbf{h}) > 0$ for all $\mathbf{h} \neq 0$. (And thus, $\mathbf{0}$ is a global minimum of Q.)

Q and H are **negative definite** if $Q(\mathbf{h}) < 0$ for all $\mathbf{h} \neq 0$.

Recall that H is positive definite if and only if all its eigenvalues are positive, and negative definite if all its eigenvalues are negative.

Example local max/min problem

Problem: Let $f(x) = \cos x + \cos y$. Classify the critical points.
Answer: First we find the critical points.

$$Df(x, y) = \begin{bmatrix} -\sin x & -\sin y \end{bmatrix}$$

So, the critical points are the points $(n\pi, m\pi)$ for integers n and m. To classify them, we look at the Hessian.

$$Hf(x, y) = \begin{bmatrix} -\cos x & 0 \\ 0 & -\cos y \end{bmatrix}$$

Example local max/min problem continued

To classify them, we look at the Hessian.

$$Hf(x,y) = \begin{bmatrix} -\cos x & 0 \\ 0 & -\cos y \end{bmatrix}$$

There are four possibilities. When both n, m are even, $Hf(n\pi, m\pi) = -I$. So in this case $(n\pi, m\pi)$ is a local maxima.

When n, m are both odd, then $Hf(n\pi, m\pi) = I$. So these points correspond to local minima.

In the mixed cases, $Hf(n\pi, m\pi) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ or $Hf(n\pi, m\pi) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. So, these are saddle points.

More topology

Let $X \subset \mathbb{R}^n$.

Definition: $\mathbf{x} \in \mathbb{R}^n$ is a **boundary point** of X if every open ball containing $\mathbf{x}$ contains both points in X and points not in X.

Definition: X is **closed** if it contains all its boundary points.

Definition: X is **bounded** if it is contained in some open ball centered at the origin.

Definition: X is **compact** if it is both closed and bounded.

Theorem

Let $X \subset \mathbb{R}^n$ be compact and $f : X \to \mathbb{R}$ be a continuous function. Then f has a global max and a global min on X. That is, there are $\mathbf{a}_{max}, \mathbf{a}_{min} \in X$ with

$$f(\mathbf{a}_{min}) \leq f(\mathbf{x}) \leq f(\mathbf{a}_{max})$$

Steps to find the global extrema

Problem: Find the global maxima and minima of a continuous function on a two dimensional compact region.

General Technique: There are typically only be a finite number of critical points on a compact region.

1. Find all critical points on the interior of the region.
2. Parameterize the boundary and find all critical points.
3. List all the corners. (Boundary points of the boundary pieces!)
4. Compare the values of the function at the list of points. Pick out the global maxima and minima.