Directional derivatives of differentiable functions

Theorem 6.2: Let $X \subset \mathbb{R}^n$ be open and suppose that $f : X \to \mathbb{R}$ is differentiable at $a \in X$. Then the directional derivative $D_{\mathbf{u}} f(a)$ exists for all directions (unit vectors) $\mathbf{u} \in \mathbb{R}^n$. Moreover,

$$D_{\mathbf{u}} f(a) = \nabla f(a) \cdot \mathbf{u}.$$

Proof: Let $h(\mathbf{x})$ be the linear approximation to f at $\mathbf{a}$. Namely,

$$h(\mathbf{x}) = \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + f(\mathbf{a}).$$

By definition of differentiability, we know that

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{f(\mathbf{x}) - h(\mathbf{x})}{\|\mathbf{x} - \mathbf{a}\|} = \lim_{\mathbf{x} \to \mathbf{a}} \frac{f(x) - \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) - f(\mathbf{a})}{\|\mathbf{x} - \mathbf{a}\|} = 0.$$

Let $\mathbf{y} = \mathbf{x} - \mathbf{a}$. Then the above limit becomes

$$\lim_{\mathbf{y} \to \mathbf{0}} \frac{f(\mathbf{a} + \mathbf{y}) - \nabla f(\mathbf{a}) \cdot \mathbf{y} - f(\mathbf{a})}{\|\mathbf{y}\|} = 0$$

From the last slide

$$\lim_{\mathbf{y} \to \mathbf{0}} \frac{f(\mathbf{a} + \mathbf{y}) - \nabla f(\mathbf{a}) \cdot \mathbf{y} - f(\mathbf{a})}{\|\mathbf{y}\|} = 0 \qquad (1)$$

Recall the definition of the directional derivative

$$D_{\mathbf{u}} f(\mathbf{a}) = \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{u}) - f(\mathbf{a})}{t}.$$

The two equations are fairly similar. In fact, we can almost get the directional derivative by restricting attention to the path $\mathbf{y} = t\mathbf{u}$. Then we have $\|\mathbf{y}\| = \|t\mathbf{u}\| = t$ when $t > 0$ and so from equation (1) we know that

$$\lim_{t \to 0^+} \frac{f(\mathbf{a} + t\mathbf{u}) - \nabla f(\mathbf{a}) \cdot (t\mathbf{u}) - f(\mathbf{a})}{t} = 0.$$

$$\lim_{t \to 0^+} \frac{f(\mathbf{a} + t\mathbf{u}) - t\nabla f(\mathbf{a}) \cdot \mathbf{u} - f(\mathbf{a})}{t} = 0.$$

$$\lim_{t \to 0^+} \frac{f(\mathbf{a} + t\mathbf{u}) - f(\mathbf{a})}{t} = \nabla f(\mathbf{a}) \cdot \mathbf{u}.$$

Similar arguments work from the left, so we have $D_{\mathbf{u}} f(a) = \nabla f(a) \cdot \mathbf{u}$.