Math290-2 Midterm I Practice Problems

Winter 2008

1. Complete the following definitions:

   (a) "An $n \times n$ matrix A is **orthogonal** if ..."

   (b) "The **angle** between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ is ..."

   (c) "A square matrix A is said to be **orthogonally diagonalizable** if ..."

   (d) "A quadratic form Q is **negative definite** if ..."

   (e) "Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** if ..."

2. True/False

   (a) If $\mathbf{x} \cdot \mathbf{y} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$ or $||\mathbf{x} - \mathbf{y}|| > ||\mathbf{y}||$.

   (b) If all the entries of the matrix of a quadratic form are positive, then the quadratic form is positive definite.

   (c) Every quadratic form is either positive semidefinite or negative semidefinite.

   (d) The determinant of every orthogonal matrix is either 1 or -1.

   (e) Every $m \times n$ matrix Q with the property that the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$ equals the angle between $\mathbf{x}$ and $\mathbf{y}$ for all nonzero $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ also has the property that $Q\mathbf{x} \cdot Q\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

   (f) The nullspace of every orthogonal matrix is $\{\mathbf{0}\}$.

   (g) Every orthogonal set of vectors in $\mathbb{R}^n$ is linearly independent.

   (h) If A is a $m \times n$ matrix, then $(\text{Nul } A)^\perp = \text{Col } A$.

   (i) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a set of vectors in $\mathbb{R}^n$, then the map

   $$p(\mathbf{y}) = \frac{\mathbf{v}_1 \cdot \mathbf{y}}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \ldots + \frac{\mathbf{v}_p \cdot \mathbf{y}}{\mathbf{v}_p \cdot \mathbf{v}_p}\mathbf{v}_p$$

   is linear.

   (j) If A is an $n \times n$ orthogonal matrix, then $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.

   (k) Powers of symmetric matrices are symmetric.

   (l) If $Q : \mathbb{R}^2 \to \mathbb{R}$ is a quadratic form with $Q(1,0) > 0$ and $Q(0,1) > 0$, then Q is positive definite.

3. Let $Q : \mathbb{R}^2 \to \mathbb{R}$ be the quadratic form defined by $Q(\mathbf{x}) = 3x_1^2 - 4x_1x_2$.

   (a) Find the matrix of Q.

   (b) Classify Q as either *positive definite*, *negative definite*, *positive semidefinite*, *negative semidefinite*, or *indefinite*.

(c) Find the principal axes of Q.

(d) On separate axes, sketch the curves described by the equations $Q(\mathbf{x}) = 0$, $Q(\mathbf{x}) = 4$, and $Q(\mathbf{x}) = -4$.

4. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$. Prove that

$$\|\mathbf{a} - \mathbf{b}\| \leq \|\mathbf{a} - \mathbf{c}\| + \|\mathbf{c} - \mathbf{b}\|.$$

5. Orthogonally diagonalize $A = \begin{bmatrix} 0 & -2 & 2 \\ -2 & 4 & -4 \\ 0 & -4 & 4 \end{bmatrix}$ if possible.

6. Let $W = \text{Span}\{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\}$.

(a) Find the standard matrix, A, for orthogonal projection onto W.

(b) What is the standard matrix, B, for the orthogonal projection onto $W^\perp$?

(c) What is AB? Why?

7. Let Q be a quadrilateral, with side lengths of l_1, l_2, l_3, and l_4 in order. Show that if $l_1^2 + l_3^2 = l_2^2 + l_4^2$ then the diagonals of Q are orthogonal.