Notes on §7.2-7.3: Vector surface integrals, Stokes's theorem, and Gauss's theorem

7.2: Vector Surface Integrals (pp. 424)

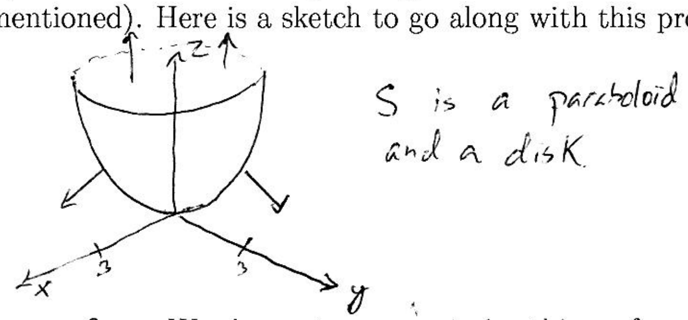
This is a review of vector surface integrals. We will just consider the following example.

Example: Consider the vector field $\mathbf{F}(x,y,z) = x\mathbf{i}+y\mathbf{j}+z^2\mathbf{k}$. Let S be the surface which is the boundary of the region $R = \{(x,y,z)\,:\,x^2+y^2 \le z \le 9\}$. Compute the flux of $\mathbf{F}$ across S.

Solution: The flux of $\mathbf{F}$ across S is the value of the integral

$$(*) = \iint_S \mathbf{F}\cdot d\mathbf{S}.$$

Since S is a closed surface, we use the outward pointing normal vector to evaluate this integral (unless otherwise mentioned). Here is a sketch to go along with this problem:



Step 1: Parameterize the surface. We choose to parameterize this surface using cylinderical coordinates, because the region looks nicest in these coordinates. We use two pieces. $\mathbf{X}_1$ will parameterize the parabolic base and $\mathbf{X}_2$ will parameterize the planar top.

$$\mathbf{X}_1(r,\theta) = (r\cos\theta, r\sin\theta, r^2) \text{ where } 0 \le r \le 3 \text{ and } 0 \le \theta \le 2\pi.$$

$$\mathbf{X}_2(r,\theta) = (r\cos\theta, r\sin\theta, 9) \text{ where } 0 \le r \le 3 \text{ and } 0 \le \theta \le 2\pi.$$

Step 2: Compute the outward pointing standard normal vectors. We will first do the $\mathbf{X}_1$ piece. We first compute the standard tangent vectors for $\mathbf{X}_1$.

$$\mathbf{T}_r = \frac{\partial}{\partial r}\mathbf{X}_1 = (\cos\theta, \sin\theta, 2r) \qquad \mathbf{T}_\theta = \tfrac{\partial}{\partial\theta}\mathbf{X}_1 = (-r\sin\theta, r\cos\theta, 0)$$

The standard normal vector $\mathbf{N}$ is either $\pm\mathbf{T}_r \times \mathbf{T}_\theta$ (with the sign determined so that the normal vector points outward). We compute

$$\mathbf{T}_r \times \mathbf{T}_\theta = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

This is the wrong sign (because the outward pointing vector should point in the downward direction, but the z coordinate of $\mathbf{T}_r \times \mathbf{T}_\theta$ is always positive). So, our standard normal vector is

$$\mathbf{N} = -\mathbf{T}_r \times \mathbf{T}_\theta = (2r^2\cos\theta, 2r^2\sin\theta, -r).$$

Now we will look at the $\mathbf{X}_2$ piece. We compute the tangent vectors, and their cross product.

$$\mathbf{T}_r = \frac{\partial}{\partial r}\mathbf{X}_2 = (\cos\theta, \sin\theta, 0) \qquad \mathbf{T}_\theta = \tfrac{\partial}{\partial\theta}\mathbf{X}_2 = (-r\sin\theta, r\cos\theta, 0)$$

$$\mathbf{T}_r \times \mathbf{T}_\theta = (0, 0, r) \qquad \mathbf{N} = \mathbf{T}_r \times \mathbf{T}_\theta = (0, 0, r)$$

(We chose $\mathbf{N} = \mathbf{T}_r \times \mathbf{T}_\theta$ rather than $\mathbf{N} = -\mathbf{T}_r \times \mathbf{T}_\theta$, so that the normal vector points upward, away from the region.)

Step 3: Evaluate the integral. The flux of $\mathbf{F}$ through S is

$$(*) = \iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iint\limits_{\mathbf{X}_1} \mathbf{F}\cdot d\mathbf{S} + \iint\limits_{\mathbf{X}_2} \mathbf{F}\cdot d\mathbf{S}.$$

We begin by evaluating the integral over $\mathbf{X}_1$.

$$\begin{aligned}
\textstyle\iint_{\mathbf{X}_1} \mathbf{F}\cdot d\mathbf{S} &= \textstyle\int_0^{2\pi}\int_0^3 \mathbf{F}(\mathbf{X}_1(r,\theta))\cdot \mathbf{N}\; dr\; d\theta \\
&= \textstyle\int_0^{2\pi}\int_0^3 (r\cos\theta, r\sin\theta, r^4)\cdot(2r^2\cos\theta, 2r^2\sin\theta, -r)\; dr\; d\theta \\
&= \textstyle\int_0^{2\pi}\int_0^3 2r^3 - r^5\; dr\; d\theta \\
&= 2\pi[\tfrac{1}{2}r^4 - \tfrac{1}{6}r^6]_{r=0}^{r=3} = -162\pi
\end{aligned}$$

Now we evaluate the integral over $\mathbf{X}_2$.

$$\begin{aligned}
\textstyle\iint_{\mathbf{X}_2} \mathbf{F}\cdot d\mathbf{S} &= \textstyle\int_0^{2\pi}\int_0^3 \mathbf{F}(\mathbf{X}_2(r,\theta))\cdot \mathbf{N}\; dr\; d\theta \\
&= \textstyle\int_0^{2\pi}\int_0^3 (r\cos\theta, r\sin\theta, 81)\cdot(0, 0, r)\; dr\; d\theta \\
&= \textstyle\int_0^{2\pi}\int_0^3 81r\; d\theta \\
&= 2\pi[\tfrac{81}{2}r^2]_{r=0}^{r=3} = 729\pi
\end{aligned}$$

So, the total flux of $\mathbf{F}$ through S is

$$(*) = \iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iint\limits_{\mathbf{X}_1} \mathbf{F}\cdot d\mathbf{S} + \iint\limits_{\mathbf{X}_2} \mathbf{F}\cdot d\mathbf{S} = 567\pi$$

7.3: Stokes's theorem (pp. 439)

Stokes's theorem is a generalization of Green's theorem to surfaces in 3-dimensional space. (Green's theorem can be derived from Stokes's theorem by considering the surface to be a subset of the plane.)

Theorem 1 (Stoke's theorem) *Let S be a bounded, piecewise smooth, oriented surface in $\mathbb{R}^3$. Suppose that ∂S consists of finitely many piecewise C^1, simple, closed curves each of which is oriented consistently with S. Let $\mathbf{F}$ be a vector field of class C^1 whose domain includes S. Then*

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Oriented consistently has a similar meaning as in Green's theorem. The choice of a normal of the surface tells which side of the surface is "up." As you walk around the boundary of the surface (standing with your head in the "up" direction), the surface should be on your left.

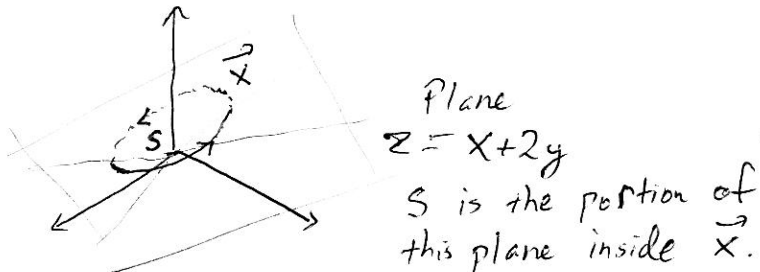
This theorem is useful to simplify many sorts of problems. Here are some types of problems you might see. (You may see others too.)

1. **Problem:** Suppose C is a simple closed curve (or collection of simple closed curves). Compute the integral $\oint_C \mathbf{F} \cdot d\mathbf{s}$.
   Possible solutions:
   (a) Direct evaluation. (This is a good method unless $\mathbf{F}$ looks too complicated.)
   (b) Find an oriented surface S such that $\partial S = C$ counting orientation. Then by Stokes's theorem,
   $$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$
   This works particularly well if $\nabla \times \mathbf{F}$ is much simpler than $\mathbf{F}$ and S is a relatively simple surface.

2. **Problem:** Compute the integral $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$.
   Possible solutions:
   (a) Direct evaluation. (Probably not the best method especially if S looks complicated.)
   (b) Direct evaluation of the other side of Stokes's theorem, $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$. (This is good if the boundary is not too complicated and $\mathbf{F}$ is not too complicated.)
   (c) Evaluate $\iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S}$, where S' is another surface with the same boundary with the same boundary orientation. This works because of two applications of Stokes's theorem.
   $$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial S'} \mathbf{F} \cdot d\mathbf{s} = \iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

Example problem: Let $\mathbf{F}(x, y, z) = (e^{x^2}, x + z, \sin z^3)$ and $\mathbf{x}(t) = (\cos t, \sin t, \cos t + 2\sin t)$. Compute $\oint_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$.

Solution: Here is a sketch to go along with this problem:



A big clue is that e^{x^2} is very difficult to integrate. We compute the curl of $\mathbf{F}$.

$$\nabla \times \mathbf{F} = (-1, 0, 1).$$

The curl of $\mathbf{F}$ is much simpler, so it would be advantageous to use Stokes's theorem. We need to find a surface with this boundary. Note that $\mathbf{x}$ lies in the plane $z = x + 2y$. Our surface S is the portion of the plane $z = x + 2y$ inside the curve $\mathbf{x}$ with upward pointing normals. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

So, we just need to evaluate this surface integral. We can parameterize S by using cylinderical coordinates.

$$\mathbf{X}(r, \theta) = \big(r\cos\theta, r\sin\theta, r(\cos\theta + 2\sin\theta)\big).$$

We compute

$$\mathbf{T}_r = \tfrac{\partial}{\partial r}\mathbf{X} = (\cos\theta, \sin\theta, \cos\theta + 2\sin\theta) \qquad \mathbf{T}_\theta = \tfrac{\partial}{\partial \theta}\mathbf{X} = (-r\sin\theta, r\cos\theta, r(2\cos\theta - \sin\theta))$$

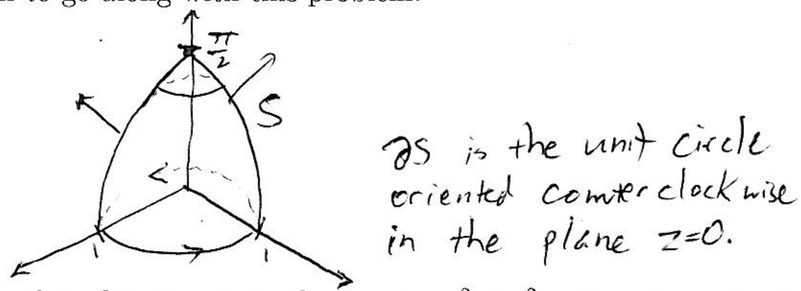
$$\mathbf{T}_r \times \mathbf{T}_\theta = (-r, -2r, r) \qquad \mathbf{N} = \mathbf{T}_r \times \mathbf{T}_\theta = (-r, -2r, r)$$

We can now evaluate.

$$\begin{aligned} \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^1 \nabla \times \mathbf{F}(\mathbf{X}(r, \theta)) \cdot \mathbf{N}\ dr\ d\theta \\ &= \int_0^{2\pi} \int_0^1 (-1, 0, 1) \cdot (-r, -2r, r)\ dr\ d\theta \\ &= \int_0^{2\pi} \int_0^1 2r\ dr\ d\theta = 2\pi \end{aligned}$$

Example problem: Let $\mathbf{F}(x, y, z) = (\frac{1}{y}\sin(x^2y^2) + x^3, \frac{1}{x}\sin(x^2y^2), 3x^2)$ and S be the surface $\cos z = \sqrt{x^2 + y^2}$ with $x^2 + y^2 \le 1$ and $0 \le z \le \frac{\pi}{2}$ oriented upward. Evaluate $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$.

Solution: Here is a sketch to go along with this problem:



The boundary of the surface S is the circle C given by $x^2 + y^2 = 1$ and $z = 0$ oriented counterclockwise. Thus, by Stokes's theorem.

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{s}.$$

However, this is not easy to evaluate either, because integrating $\mathbf{F}$ seems difficult. Let S' be the unit disk in the plane $z = 0$ with upward pointing normals. Then $\partial S' = C$. Then, by a second application of Stokes's theorem, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

We can parameterize $\mathbf{S}'$ using cylinderical coordinates.

$$\mathbf{X}(r, \theta) = (r\cos\theta, r\sin\theta, 0) \text{ with } 0 \le \theta \le 2\pi \text{ and } 0 \le r \le 1.$$

Then the standard upward pointing normal is $\mathbf{N} = (0, 0, r)$. We compute $\nabla \times \mathbf{F} = (0, -6x, -3x^2)$. Thus,

$$\begin{aligned} \iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^1 (0, -6r\cos\theta, -3r^2\cos^2\theta) \cdot (0, 0, r)\ dr\ d\theta \\ &= \int_0^{2\pi} \int_0^1 -3r^3 \cos^2\theta\ dr\ d\theta \\ &= \int_0^{2\pi} \frac{-3}{4} \cos^2\theta\ d\theta \\ &= \frac{-3}{4} \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\ d\theta = \frac{-3\pi}{4}. \end{aligned}$$

7.3: Gauss's theorem (pp. 442)

Recall that if $\mathbf{F}$ is a vector field determining the velocity of particles (perhaps in a fluid), then the divergence $\nabla \cdot \mathbf{F}$ measures the expansion or contraction of that fluid in time. The expansion of the fluid in a fixed region R would mean that the some of the fluid would have to leave the region R through the boundary. Thus, we expect that positive divergence on a region R should indicate particles are leaving through the boundary of the region R. Using more mathematical terminology, positive divergence on R indicates positive flux through the boundary ∂R when oriented with outward pointing normals. Gauss's theorem gives a much more precise description of the relationship between divergence in a region R and the flux through its boundary.

Theorem 2 (Gauss's theorem) *Let D be a bounded solid region in $\mathbb{R}^3$ whose boundary ∂D consists of finitely many piecewise smooth, closed orientable surfaces, each of which is oriented by unit normals that point* away *from D. Let $\mathbf{F}$ be a vector field of class C^1 whose domain includes D. Then*

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F} \, dV.$$

Generally (but not always), the right side of this equation is the easiest to evaluate. Triple integrals generally require less work than vector surface integrals. Also, $\nabla \cdot \mathbf{F}$ is often much simpler than $\mathbf{F}$.

Here are two types of problems you might see. (You may see others too.)

1. **Problem:** Suppose S is a oriented closed surface (or collection of such surfaces). Compute the integral $\oiint_S \mathbf{F} \cdot d\mathbf{S}$.
   Possible solutions:
   (a) Direct evaluation. (This is probably not best unless the integral looks particularly simple.)
   (b) Find a region $D \subset \mathbb{R}^3$ such that $\partial D = S$. (Keep track of orientation, because the opposite orientation will introduce a minus sign into your calculations.) Then evaluate the right hand side of Gauss's theorem, $\iiint_D \nabla \cdot \mathbf{F} \, dV$.

2. **Problem:** Suppose S is a oriented surface which is not closed. Compute the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.
   Possible solutions:
   (a) Direct evaluation. (This is probably best unless the integral seems difficult.)
   (b) Find a region $D \subset \mathbb{R}^3$ such that the boundary ∂D has S as one of its boundary components. Let S' be the remaining part of ∂D oriented outward. We should choose D so that S' is as simple as possible. Then Gauss's theorem reads

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} + \oiint_{S'} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F} \, dV.$$

(Assuming S is oriented with outward pointing normals. Otherwise introduce a minus sign in front of the integral over S.) Solving for the integral over S we get

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \nabla\cdot\mathbf{F}\ dV - \oiint_{S'} \mathbf{F}\cdot d\mathbf{S}.$$

Now evaluate the right hand side.

Example problem: (Repeat of the example on the first page) Consider the vector field $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$. Let S be the surface which is the boundary of the region $R = \{(x,y,z) : x^2+y^2 \le z \le 9\}$. Compute the flux of $\mathbf{F}$ across S.

Solution: We now use Gauss's theorem. The flux of $\mathbf{F}$ across S is the value of the integral

$$\iint_S \mathbf{F}\cdot d\mathbf{S}.$$

By Gauss's theorem, this is the same as evaluating

$$\iiint_R \nabla\cdot\mathbf{F}\ dV.$$

We compute $\nabla\cdot\mathbf{F} = 2+2z$. Now we parameterize our region in cylinderical coordinates, because the region looks nicer in this way. R is determined in cylinderical coordinates by the inequalities

$$0\le\theta\le 2\pi \quad 0\le r\le 3 \quad r^2\le z\le 9.$$

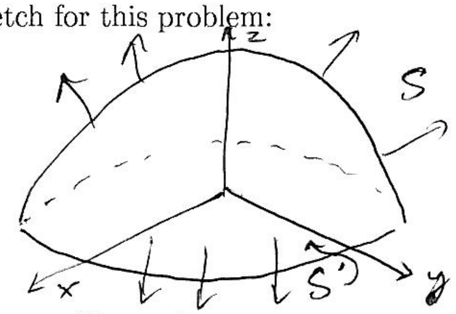
In cylinderical coordinates $dV = r\ dz\ dr\ d\theta$. Thus, the triple integral can be rewritten as

$$\begin{aligned}\int_0^{2\pi}\int_0^3\int_{r^2}^9 (2+2z)r\ dz\ dr\ d\theta &= 2\pi\int_0^3 r[2z+z^2]_{z=r^2}^{z=9}\ dr\\ &= 2\pi\int_0^3 r(99-2r^2-r^4)\ dr\\ &= 2\pi\int_0^3 99r-2r^3-r^5\ dr\\ &= 2\pi[\tfrac{99}{2}r^2-\tfrac{1}{2}r^4-\tfrac{1}{6}r^6]_{r=0}^{r=3}\\ &= \pi(891-81-243) = 567\pi\end{aligned}$$

This agrees with our answer from before, but was quite a bit easier.

Example problem: Let S be the top half of the unit sphere in $\mathbb{R}^3$ oriented with upward pointing normal vectors. Let $\mathbf{F}(x,y,z) = (e^{y^2}, 0, x^2z+y^2z+\frac{1}{3}z^3)$. Compute the flux of $\mathbf{F}$ through S.

Solution: Here is a sketch for this problem:



We are asked to compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$. But, this looks difficult, because of the e^{y^2} in $\mathbf{F}$.

Instead, let

$$D = \{(x, y, z) \text{ such that } x^2 + y^2 + z^2 \leq 1 \text{ and } z \geq 0\}.$$

Then by Gauss's theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F} \, dV - \iint_{S'} \mathbf{F} \cdot d\mathbf{S},$$

where S' is the remaining boundary component of ∂D with outward pointed normals. That is, D is the unit disk in the plane $z = 0$ with downward pointed normals.

Let us evaluate the triple integral first. We have that $\nabla \cdot \mathbf{F} = x^2 + y^2 + z^2$. It seems wise to use spherical coordinates. We can parameterize the region D in spherical coordinates by

$$0 \leq \theta \leq 2\pi \quad 0 \leq \rho \leq 1 \quad 0 \leq \phi \leq \frac{\pi}{2}$$

In spherical coordinates $dV = \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$. We evaluate the integral.

$$\begin{aligned} \iiint_D \nabla \cdot \mathbf{F} \, dV &= \int_0^{2\pi} \int_0^1 \int_0^{\frac{\pi}{2}} (\rho^2)\rho^2 \sin\phi \, d\phi \, d\rho \, d\theta \\ &= 2\pi \int_0^1 \rho^4 [-\cos\phi]_{\phi=0}^{\phi=\frac{\pi}{2}} \, d\rho \\ &= 2\pi \int_0^1 \rho^4 \, d\rho = 2\pi [\tfrac{1}{5}\rho^5]_{\rho=0}^{\rho=1} = \tfrac{2\pi}{5} \end{aligned}$$

Now, we have to evaluate the surface integral over S'. We parameterize S' with cylinderical coordinates.

$$\mathbf{X}(r, \theta) = (r\cos\theta, r\sin\theta, 0) \text{ where } 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi.$$

Now we compute the standard normal vector.

$$\mathbf{T}_r = \frac{\partial}{\partial r}\mathbf{X} = (\cos\theta, \sin\theta, 0) \qquad \mathbf{T}_\theta = \tfrac{\partial}{\partial\theta}\mathbf{X} = (-r\sin\theta, r\cos\theta, 0)$$

$$\mathbf{T}_r \times \mathbf{T}_\theta = (0, 0, r) \qquad \mathbf{N} = -\mathbf{T}_r \times \mathbf{T}_\theta = (0, 0, -r)$$

We chose $\mathbf{N} = -\mathbf{T}_r \times \mathbf{T}_\theta$ rather than $\mathbf{N} = \mathbf{T}_r \times \mathbf{T}_\theta$ so that the surface is oriented downward as required. Then we can evaluate the integral over S'.

$$\begin{aligned} \iint_{S'} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^1 \mathbf{F}(\mathbf{X}(r, \theta) \cdot \mathbf{N} \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (e^{r^2 \sin^2\theta}, 0, 0) \cdot (0, 0, -r) \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 0 \, dr \, d\theta = 0 \end{aligned}$$

By Gauss's theorem, we know the integral over S.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F} \, dV - \iint_{S'} \mathbf{F} \cdot d\mathbf{S} = \frac{2\pi}{5} - 0 = \frac{2\pi}{5}$$