Math 290-3: Practice problems for the first midterm

Remarks on the midterm

- As on the first midterm, definitions will be graded with only limited partial credit, so most definitions will receive either full or no credit. However, to make studying easier, we have provided a list of possible definitions that may appear on this midterm. (No other definitions will appear.)
  - "The **length** of a C^1 path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is ..."
  - "The **unit tangent vector T** of a C^1 path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is..."
  - "The **curvature** κ of a C^2 path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is..."
  - "A **vector field** on $X \subseteq \mathbb{R}^n$ is ..."
  - "A scalar function $f : \mathbb{R}^n \to \mathbb{R}$ is called a **potential function** of a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ if ..."
  - "A **flow line** of a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is..."
  - "The **divergence** of a C^1 vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is..."
  - "A C^1 vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is called **incompressible** if..."
  - "The **curl** of a C^1 vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is..."
  - "A C^1 vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is called **irrotational** if..."
  - "Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a piecewise C^1 path. We say that another C^1 path $\mathbf{y} : [c, d] \to \mathbb{R}^n$ is a **reparameterization** of $\mathbf{x}$ if..."
  - "Let $U \subseteq \mathbb{R}^n$ be open. A vector field $\mathbf{F} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is **conservative** if ..."
  - "A continuous vector field **F has path-independent line integrals** if..."
  - "A region R in $\mathbb{R}^2$ or $\mathbb{R}^3$ is **simply-connected** if ..."
- You will also be required to know the statements of a main results. Statements of the following may be required for the midterm.
  - Theorem 1.4 (pp. 370): The effect of reparameterization on scalar line integrals.
  - Theorem 1.5 (pp. 371): The effect of reparameterization on vector line integrals.
  - Green's Theorem (pp. 381).
  - The divergence theorem in the plane (pp. 384).
  - Theorem 3.3 (pp. 392): Equivalence of a vector field being conservative and having path-independent line integrals.
  - Theorem 3.5 (pp. 393): Necessary and sufficient conditions for a vector field in a simply connected region to be conservative.

Practice problems for the midterm

For additional practice problems, consider the exercises in §3.5-3.6 and §6.4-6.5 of the book.

1. Evaluate the vector line integral

$$\oint_C 5x^2y^3\ dx - 3x^5\ dy,$$

where C is the unit circle oriented counter-clockwise.

2. Consider the vector field $\mathbf{F}(x,y) = (2x-2y, 2y-2x)$.

   (a) Is $\mathbf{F}$ conservative? If it is conservative, find a potential function f. If it is not conservative, demonstrate that it is not.

   (b) Find the set of all points (x,y) such that there is a C^1 curve C starting at $(0,0)$ and ending at (x,y) so that

$$\int_C \mathbf{F}\cdot d\mathbf{s} = 4.$$

3. A fence is built along the curve $y = x^2$ with $1 \leq x \leq 4$ with height $h(x,y) = x$ at the point (x,y). Find the area of this fence.

4. Consider the vector field

$$\mathbf{F}(x,y) = (xe^{x^2+y^2} + 4xy)\mathbf{i} + (ye^{x^2+y^2} + 2x^2 + \sin y)\mathbf{j}.$$

   (a) Is $\mathbf{F}$ conservative? If it is conservative, find a potential function f. If it is not conservative, demonstrate that it is not.

   (b) Compute the integral

$$\oint_C \mathbf{F}\cdot d\mathbf{s}$$

   where C is the unit circle oriented counter clockwise.

5. Let $\mathbf{F} = (\frac{1}{x}e^{\frac{y^2}{x^2}}, -\frac{1}{y}e^{\frac{y^2}{x^2}})$.

   (a) Show that $\mathbf{F}$ has path independent line integrals over the region R where $x > 0$ and $y > 0$.

   (b) Consider the curve $\mathbf{x}(t) = (2t+2, t^2+1)$ for $0 \leq t \leq 1$. Compute the vector line integral

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s}.$$

   (*Hint:* This integral will be difficult to evaluate directly.)

6. Consider the curve $\mathbf{x}(t) = (\frac{t}{1+t^2}, \frac{1}{1+t^2})$.

   (a) Compute the velocity and speed of $\mathbf{x}$.

   (b) Compute the unit tangent vector $\mathbf{T}(t)$.

   (c) Compute the unit normal vector $\mathbf{N}(t)$.

   (d) Compute the curvature of this curve.

   (e) What is this curve?