Math 290-3: Preparation for the final exam

Topics on the Final

The following sections of Colley are covered by the final.

- The material from the first midterm (§5.1-5.5)
- The material from the second midterm (§3.2-3.4 and 6.1-6.3)
- Chapter 7, §7.1-7.3.
  - From §7.3 you only need to know pages 439-444.

Definitions on the final

As on the midterms this quarter, definitions will be graded with only limited partial credit. So, most definitions will receive either full or no credit. We have provided a list of possible definitions that may appear on this final. (No other definitions will appear.)

- Definitions for the first midterm.
  - "Given a closed rectangle $R = [a, b] \times [c, d]$, a **partition of** R **of order** n is ..."
  - "Given a function f from $R = [a, b] \times [c, d]$ to $\mathbb{R}$, a **Riemann sum** of f on R is ..."
  - "Let R be a closed rectangle in $\mathbb{R}^2$. If $f : R \to \mathbb{R}$ is integrable on R, then the **double integral** of f on R is ..."
  - "Given a closed box $B = [a, b] \times [c, d] \times [p, q]$, a **partition of** B **of order** n is ..."
  - "Given a function f from $B = [a, b] \times [c, d] \times [p, q]$ to $\mathbb{R}$, a **Riemann sum** of f on B is ..."
  - "Let B be a closed box in $\mathbb{R}^3$. If $f : B \to \mathbb{R}$ is integrable on B, then the **triple integral** of f on B is ..."
  - "Let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable. The **Jacobian** of $\mathbf{T}$ is ..."
- Definitions for the second midterm.
  - "The **length** of a C^1 path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is ..."
  - "The **unit tangent vector T** of a C^1 path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is..."
  - "The **curvature** κ of a C^2 path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is..."
  - "A **vector field** on $X \subseteq \mathbb{R}^n$ is ..."
  - "A scalar function $f : \mathbb{R}^n \to \mathbb{R}$ is called a **potential function** of a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ if ..."
  - "A **flow line** of a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is..."
  - "The **divergence** of a C^1 vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is..."
  - "A C^1 vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is called **incompressible** if..."

- “The **curl** of a C^1 vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is. . . ”
- “A C^1 vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is called **irrotational** if. . . ”
- “Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a piecewise C^1 path. We say that another C^1 path $\mathbf{y} : [c, d] \to \mathbb{R}^n$ is a **reparameterization** of $\mathbf{x}$ if. . . ”
- “Let $U \subseteq \mathbb{R}^n$ be open. A vector field $\mathbf{F} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is **conservative** if . . . ”
- “A continuous vector field **F has path-independent line integrals** if. . . ”
- “A region R in $\mathbb{R}^2$ or $\mathbb{R}^3$ is **simply-connected** if . . . ”

- Definitions from chapter 7.
  - “A **parameterized surface** in $\mathbb{R}^3$ is . . . ”
  - “The parameterized surface $S = \mathbf{X}(D)$ is **smooth** at $\mathbf{X}(s_0, t_0)$ if . . . ”
  - “Let D be a bounded subset of the st-plane and $\mathbf{X} : D \to \mathbb{R}^3$. The **surface area** of a smooth parameterized surface $S = \mathbf{X}(D)$ is . . . ”
  - “Let $\mathbf{X} : D \to \mathbb{R}^3$ be a parameterized surface, where D is a bounded subset of the st-plane. Let f be a continuous function whose domain includes $S = \mathbf{X}(D)$. Then the **scalar surface integral** of f along $\mathbf{X}$ is . . . ”
  - “Let $\mathbf{X} : D \to \mathbb{R}^3$ be a parameterized surface, where D is a bounded subset of the st-plane. Let $\mathbf{F}$ be a continuous vector field whose domain includes $S = \mathbf{X}(D)$. Then the **vector surface integral** of $\mathbf{F}$ along $\mathbf{X}$ is . . . ”
  - “Let $S = \mathbf{X}(D)$ be a parameterized surface, and $\mathbf{F}$ be a continuous vector field whose domain includes S. The **flux of F across** S is. . . ”
  - “Let $\mathbf{X} : D_1 \subset \mathbb{R}^2 \to \mathbb{R}^3$ and $\mathbf{Y} : D_1 \subset \mathbb{R}^2 \to \mathbb{R}^3$ be parameterized surfaces. We say that $\mathbf{Y}$ is a **reparameterization** of $\mathbf{X}$ if. . . ”
  - “A smooth, connected surface S is **orientable** if. . . ”

Theorems on the final

You will also be required to know the statements of a few main results. Statements of the following may be required for the final.

- Theorems from the first midterm
  - Proposition 1.1 (pp. 291): “The volume under the graph of a continuous non-negative function $f : [a, b] \times [c, d] \to \mathbb{R}$ and over the rectangle $[a, b] \times [c, d]$ is . . . ”
  - Theorem 2.5 (pp. 295): Sufficient conditions for integrability.
  - Theorem 2.6 (pp. 296): Fubini’s theorem for double integrals.
  - Theorem 5.3 (pp. 329): The change of variables theorem in double integrals.

- Theorems from the second midterm.
  - Theorem 1.4 (pp. 370): The effect of reparameterization on scalar line integrals.

- Theorem 1.5 (pp. 371): The effect of reparameterization on vector line integrals.
- Green's Theorem (pp. 381).
- The divergence theorem in the plane (pp. 384).
- Theorem 3.3 (pp. 392): Equivalence of a vector field being conservative and having path-independent line integrals.
- Theorem 3.5 (pp. 393): Necessary and sufficient conditions for a vector field in a simply connected region to be conservative.

• Theorems from chapter 7.

- Theorem 2.5 (pp. 428): The effect of reparameterization on vector surface integrals.
- Stokes's theorem (pp. 439)
- Gauss's theorem (pp. 442)

Practice problems

These problems emphasize material covered since the second midterm. However, the final will test all material covered in the course. For practice problems for earlier parts of the course, you are encouraged to revisit the previous practice problems for the midterms and your actual midterms. For additional practice problems, consider the exercises in §3.5-3.6, §5.7-5.8, §6.4-6.5 and §7.5-7.6 of the book.

1. Let S be the portion of the parabolid $z = x^2 + y^2$ inside the cylinder $(x-1)^2 + y^2 = 1$. Orient S with upward pointing normals.
   (a) Use modified cylindrical coordinates (centered at the point $(1, 0, 0)$) to parameterize S.
   (b) Directly compute the flux of the vector field $\mathbf{F}(x, y, z) = (1, 1, 2y)$ through S.
2. Suppose R is a bounded region in $\mathbb{R}^3$ and $S = \partial R$ is oriented with normals pointing away from R. Let $\mathbf{F}(x, y, z) = (x, y, z)$. Show that
$$\oiint_S \mathbf{F} \cdot d\mathbf{S}$$
   is three times the volume of R.
3. Show that Green's theorem is a special case of Stokes's theorem.
4. Let S be the ellipsoid S determined by the equation
$$2x^2 + 3y^2 + 5z^2 = 30.$$
   Find a formula for the outward pointing unit normal vector to S at each point (x, y, z) on S.

5. Let R be a bounded region in $\mathbb{R}^3$, and orient $S = \partial R$ with normal vectors pointing away from R. Suppose that $\mathbf{F}$ and $\mathbf{G}$ are C^1 vector fields defined on a region containing R. Prove that if $\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{G}$ then

$$\iint_{\partial R} \mathbf{F} \cdot d\mathbf{S} = \iint_{\partial R} \mathbf{G} \cdot d\mathbf{S}.$$

6. Evaluate the integral

$$\int_C (y + \sin x)\ dx + (z^2 + \cos y)\ dy + x^3\ dz$$

where C is the curve $\mathbf{x}(t) = (\sin t, \cos t, \sin 2t)$ for $0 \leq t \leq 2\pi$. (*Hint:* Observe that C lies on the surface $z = 2xy$.)

7. Consider the curve C given in polar coordinates by the equation $r = 1 + \sin\theta$ with $0 \leq \theta \leq \pi$. Use Green's theorem to compute

$$\int_C 3x^2\ dx + (x + y + \tan^{-1}(y^{43}))\ dy.$$

(*Hint:* The curve C is not closed. You will need to find a region R which contains C as one of its boundary components. Then apply Green's theorem to this region.)

8. Let $\mathbf{F}(x, y, z) = z \tan^{-1}(y^2)\mathbf{i} + z^3 \ln(x^2+1)\mathbf{j} + z\mathbf{k}$. Let S be the portion of the paraboloid $z = x^2 + y^2 - 2$ with $z \leq 1$ oriented upward. Find the flux of $\mathbf{F}$ across S. (*Hint:* Find a region R with S a piece of the boundary. Then apply the divergence theorem.)

9. Let $D = \{(x, y)\ :\ 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$. Suppose that the scalar function $f : D \to \mathbb{R}$ is of class C^2. Find a formula for the surface area of the graph $z = f(x, y)$.