M.E.N.U. Linear algebra: Practice Midterm II

November 2007

1. Complete the following definitions:

   (a) "A function $T : \mathbf{R}^n \to \mathbf{R}^m$ is a *linear transformation* if..."

   (b) "Let H be a subspace of $\mathbf{R}^n$. A collection of vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ in H is a *basis* for H if..."

   (c) "The *null space* of an $m \times n$ matrix A is defined:

   $$\mathrm{Nul}(A) = \dots\text{"}$$

   (d) "The *column space* of an $m \times n$ matrix A is defined:

   $$\mathrm{Col}(A) = \dots\text{"}$$

2. Let S and T be linear transformations of $\mathbf{R}^2$ into itself such that

   $$S\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad S\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix},$$

   and

   $$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

   (a) Find the standard matrix for T.

   (b) Find the standard matrix for the composition $S \circ T$, where $S \circ T(\mathbf{x}) = S(T(\mathbf{x}))$.

3. Let

   $$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 3 \\ -2 & -4 & 4 \end{pmatrix}$$

   (a) Find a basis for $\mathrm{Col}(A)$.

   (b) Find a basis for $\mathrm{Nul}(A)$.

   (c) Is A invertible? Justify your answer.

4. Let

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -3 \\ 1 & 2 \end{pmatrix}$$

Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation given by the formula $T(\mathbf{x}) = A\mathbf{x}$.

(a) Let $\mathbf{b} = \begin{pmatrix} 5 \\ -2 \\ 13 \end{pmatrix}$. Does there exist a vector $\mathbf{v}$ in $\mathbf{R}^2$ such that $T(\mathbf{v}) = \mathbf{b}$? If so, find such a vector, if not, explain why such a vector does not exist.

(b) Is T one-one? Onto? Justify your answers.

5. Is there a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ with the properties:

$$T(1,0) = (2,-1,0), \quad T(0,1) = (1,1,3), \quad T(1,-1) = (1,-2,-1)?$$

If so, give a formula for T. If not, explain why such a T cannot exist.

6. Consider two 2×2 matrices, A and B. Suppose that

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \text{and} \qquad (AB)^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

Find A^{-1}. **Justify the steps in your calculation.**