1. Do the following limits exist? If they do exist, find the limit. If they do not, explain why they do not.

(a) *(10 points)*

$$\lim_{(x,y)\to(0,0)} \frac{|xy|}{|x|+|y|}$$

$$= \lim_{r\to 0} \frac{r^2|\cos\theta\sin\theta|}{r(|\cos\theta|+|\sin\theta|)} = \lim_{r\to 0} \frac{r|\cos\theta\sin\theta|}{|\cos\theta|+|\sin\theta|} = \boxed{0}$$

because $a = \frac{|\cos\theta\sin\theta|}{|\cos\theta|+|\sin\theta|}$ is bounded independent from θ.

Note: $\frac{1}{a} = \frac{1}{|\sin\theta|} + \frac{1}{|\cos\theta|}$ so $\frac{1}{a} \geq 2$ thus $a \leq \frac{1}{2}$.

(b) *(10 points)*

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^4}{(x+y)^4}$$

Does not exist

Consider path $x(t) = t$ $y(t) = 0$

$$\lim_{t\to 0} \frac{(t-0)^4}{t^4} = 1$$

Now consider path $x(t) = t$ $y(t) = t$

$$\lim_{t\to 0} \frac{(t-t)^4}{(t+t)^4} = \lim_{t\to 0} \frac{0}{(2t)^4} = 0.$$

2. The function $z(x, y)$ is defined implicitly by the equation

$$F = \sin(x+z) - ye^z = 0$$

(a) *(10 points)* Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by implicit differentiation.

(b) *(10 points)* Find an equation for the tangent plane to $z(x, y)$ at the point $(\frac{\pi}{2}, 1, 0)$.

a) $F_x = \cos(x+z)$

$F_y = -e^z$

$F_z = \cos(x+z) - ye^z$

So $$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-\cos(x+z)}{\cos(x+z) - ye^z}$$

$$\frac{\partial z}{\partial y} = \frac{+e^z}{\cos(x+z) - ye^z}$$

b) $\frac{\partial z}{\partial x}(\frac{\pi}{2}, 1, 0) = 0$

$\frac{\partial z}{\partial y}(\frac{\pi}{2}, 1, 0) = -1$

$$z = \frac{\partial z}{\partial x}(x - \frac{\pi}{2}) + \frac{\partial z}{\partial y}(y-1) + 0$$

$$\boxed{z = 1 - y.}$$

3. The position of a pigeon at time t is given by

$$\mathbf{r}(t) = \langle \cos t, \sin t, \frac{1}{3}(2+t^2)^{\frac{3}{2}} \rangle$$

The pigeon's velocity and acceleration are given by

$$\mathbf{v}(t) = \langle -\sin t, \cos t, t\sqrt{2+t^2} \rangle \quad \text{and} \quad \mathbf{a}(t) = \langle -\cos t, -\sin t, \frac{2+2t^2}{\sqrt{2+t^2}} \rangle$$

(a) *(5 points)* Find the speed of the pigeon at time t.

(b) *(5 points)* Find the tangential component of acceleration of the pigeon at time t.

(c) *(5 points)* Find the curvature of $\mathbf{r}(t)$ at time $t = 0$.

(d) *(5 points)* Find the normal component of acceleration of the pigeon at time $t = 0$.

a) $\text{speed} = \|v(t)\| = \sqrt{\sin^2 t + \cos^2 t + t^2(2+t^2)}$

$\frac{ds}{dt} = \sqrt{1 + 2t^2 + t^4} = \sqrt{(1+t^2)^2}$

$\frac{ds}{dt} = \boxed{1+t^2}$ since $1+t^2 > 0$.

b) $a_T = \frac{d}{dt}\left[\frac{ds}{dt}\right] = \frac{d}{dt}\left[1+t^2\right] = \boxed{2t.}$

c) $K = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{\|\langle 0,1,0\rangle \times \langle -1, 0, \sqrt{2}\rangle\|}{\|\langle 0,1,0\rangle\|^3}$

$K = \frac{\sqrt{3}}{1} = \boxed{\sqrt{3}}$

d) $\|\vec{a}(0)\| = \|\langle -1, 0, \sqrt{2}\rangle\| = \sqrt{3}$

$a_N = \sqrt{\|\vec{a}(0)\|^2 - a_T^2} = \sqrt{(\sqrt{3})^2 - 0} = \boxed{\sqrt{3.}}$

4. Consider the surface S described parametrically by the following equations

$$x(t,u) = e^{t^2+u}\cos t \qquad y(t,u) = e^{t^2+u}\sin t \qquad z(t,u) = t^2 + u$$

(a) *(5 points)* Find a plane containing the points $\big(x(0,u), y(0,u), z(0,u)\big)$ for all u.

(b) *(5 points)* Find an equation for the intersection between S and the plane $z = 0$. Describe this intersection in words.

(c) *(5 points)* Do the same for the intersection between S and the plane $z = k$ for all real numbers k.

(d) *(5 points)* Using cylindrical coordinates, find a new parametric representation for S.

a) $x(0,u) = e^u \qquad y(0,u) = 0 \qquad z(0,u) = u.$

Plane is $\boxed{y=0}$

b) if $z=0$ then $t^2+u=0$ so $u=-t^2$.

$x(t,-t^2) = e^0 \cos t = \cos(t)$

$y(t,-t^2) = e^0 \sin t = \sin(t)$

So $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = 0$ is a parametric equation for the intersection, which is the unit circle in the xy-plane.

c) $z=K$ then $t^2+u=K$ so $u = K-t^2$.

$x(t, K-t^2) = e^K \cos t$

$y(t, K-t^2) = e^K \sin t$

} circle of radius e^K in plane $z=K$.

d) $x(z,\theta) = e^z \cos\theta$

$y(z,\theta) = e^z \sin\theta$

$z(z,\theta) = z$

5. Suppose that a penguin is climbing an iceberg. The iceberg can be described by graphing the differentiable function $f(x, y)$. At time t the position of the penguin is given by

$$\mathbf{r}(t) = \langle x(t), y(t), f(x(t), y(t)) \rangle$$

Suppose that the tangent plane to $f(x, y)$ at the point where the penguin is standing at time $t = 1$ is given by the formula

$$2x - 3y - z = 15$$

Suppose also that $\frac{dx}{dt}(1) = 2$ and $\frac{dy}{dt}(1) = \pi$.

(a) *(10 points)* What are $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(x(1), y(1))$?

(b) *(10 points)* What is $\frac{df}{dt}$ at time $t = 1$? Is the altitude of the penguin increasing or decreasing?

a) The tangent plane at this point is

$$z = 2x - 3y - 15$$

So of course this is the same as

$$z = f_x(x(1), y(1))(x-a) + f_y(x(1), y(1))(y-b) + f(x(1), y(1))$$

where $a = x(1)$ $b = y(1)$

So $f_x(x(1), y(1)) = 2$

$f_y(x(1), y(1)) = -3$

b) $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{df}{dt}(0) = 2(2) + (-3)\pi = 4 - 3\pi$$

Since $4 - 3\pi < 0$, the penguin's altitude is decreasing.