1. Let $\mathcal{P}$ be the plane $x + y + z = 3$.

   (a) *(6 points)* Find a normal vector to $\mathcal{P}$.

   (b) *(7 points)* Does the line parameterized by

   $$x(t) = 3 + t \quad y(t) = 2t - 2 \quad z(t) = -t$$

   intersect this plane? If it does, find the intersection point.

   (c) *(7 points)* Find the (shortest) distance between the point $Q = (3, 4, 5)$ and the plane $\mathcal{P}$.

a) $\vec{n} = \langle 1, 1, 1 \rangle$

b) By substitution,

$$(3+t) + (2t-2) + (-t) = 3$$
$$2t + 1 = 3$$
$$t = 1$$

so $P = (x(1), y(1), z(1))$
$= (4, 0, -1)$

c) $\text{Dist}(\mathcal{P}, Q) = \left| \text{comp}_{\vec{n}} \overrightarrow{PQ} \right| = \left| \text{comp}_{\langle 1,1,1 \rangle} \langle -1, 4, 6 \rangle \right|$

$$= \left| \frac{\langle -1, 4, 6 \rangle \cdot \langle 1, 1, 1 \rangle}{\| \langle 1, 1, 1 \rangle \|} \right|$$

$$= \frac{9}{\sqrt{3}} = \boxed{3\sqrt{3}}$$

2. Suppose that an spacecraft follows the path

$$\mathbf{r}(t) = \langle 2t\cos t, 2t\sin t, \frac{1}{3}t^3 \rangle$$

(a) *(10 points)* Find the derivative $\mathbf{r}'(t)$. This is the velocity vector of the spacecraft at time t. (*Hint:* If computed correctly, $\| \mathbf{r}'(t) \| = t^2 + 2$. The quantity $\| \mathbf{r}'(t) \|$ is the speed of the spacecraft at time t.)

(b) *(10 points)* Find the arc length of $\mathbf{r}(t)$ with $0 \le t \le 3$. This is the distance traveled by the spacecraft between time 0 and time 3.

a) $r'(t) = \langle 2\cos t - 2t\sin t, 2\sin t + 2t\cos t, t^2 \rangle$

note $\| r'(t) \| = \sqrt{(2\cos t - 2t\sin t)^2 + (2\sin t + 2t\cos t)^2 + (t^2)^2}$

$= \sqrt{4\cos^2 t + 4t^2\sin^2 t + 4\sin^2 t + 4t^2\cos^2 t + t^4}$

$= \sqrt{4 + 4t^2 + t^4}$

$= \sqrt{(t^2+2)^2} = t^2 + 2.$

b) Arclength $= \int_0^3 \| r'(t) \| dt$

$= \int_0^3 t^2 + 2 \, dt = \left[\frac{1}{3}t^3 + 2t \right]_0^3$

$= 9 + 6 = 15$

3. Let $\mathcal{H}$ be the hyperboloid of one sheet determined by the equation

$$x^2 + y^2 - z^2 = 1$$

(a) *(7 points)* Draw the trace of the hyperboloid $\mathcal{H}$ in the plane $x = -1$. (This can be drawn in the plane $x = -1$.)

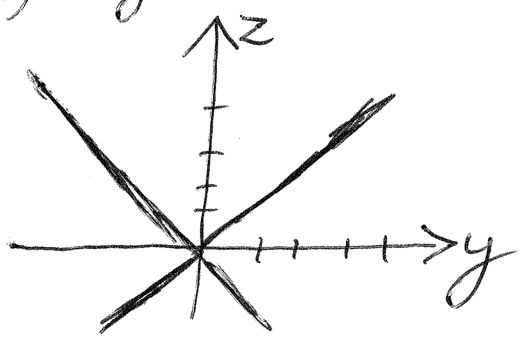
(b) *(7 points)* Find an equation for the plane $\mathcal{P}$ containing the point $Q = (1, 1, 1)$ with normal vector $\mathbf{n} = \langle 1, 1, -1 \rangle$.

(c) *(6 points)* Find a parameterization for a curve contained in the intersection between $\mathcal{P}$ and $\mathcal{H}$. (*Hint:* The intersection looks like the solution to part (a).)

a) By substitution $(-1)^2 + y^2 - z^2 = 1$

$y^2 = z^2$

$y = \pm z$



b) $(x-1) + (y-1) - (z-1) = 0$

$x + y - z = 1$

c) On $\mathcal{P}$, $z = -1 + x + y$.

By substitution, $x^2 + y^2 - (-1 + x + y)^2 = 1$

$x^2 + y^2 - 1 - x^2 - y^2 + 2x + 2y - 2xy = 1$

$0 = 2xy - 2x - 2y + 2$

$0 = xy - x - y + 1$

$0 = (x-1)(y-1)$ So either $x = 1$ or $y = 1$

If $x = 1$, $z = +y$, so $x(t) = 1$, $y(t) = t$, $z(t) = +t$ or If $y = 1$, $z = x$ $x(t) = t$ $y(t) = 1$ $z(t) = t$

4. True or False? In mathematics, a statement is true if it is true for all possible variable values, and false otherwise.

   (a) *(4 points)* If $\| \mathbf{a} \times \mathbf{b} \| = 0$, then $\mathbf{a}$ is orthogonal to $\mathbf{b}$.

   False, if $\|a \times b\| = 0$ then $\vec{a}$ and $\vec{b}$ are parallel.

   (b) *(4 points)* Pairs of lines in 3 dimensional space either intersect or are parallel.

   False, they might be skew!

   (c) *(4 points)* The vector projection of $\mathbf{a}$ onto $\mathbf{b}$ is always shorter than $\mathbf{a}$.

   Ambiguous. $\|\mathrm{proj}_{\vec{b}} \vec{a}\| \leq \|\vec{a}\|$.
   (points for either answer).

   (d) *(4 points)* $|\mathbf{a} \cdot \mathbf{b}| \leq \| \mathbf{a} \| \| \mathbf{b} \|$.

   True $|a \cdot b| = \|a\| \|b\| \cos\theta \leq \|a\| \|b\|$.

   (e) *(4 points)* $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

   True. This is the volume of a flat parallellpipped

5. Consider the ellipse $\mathcal{E}$ in the plane determined by the equation

$$x^2 - 4x + 4(y-1)^2 = 1$$

(a) *(5 points)* Find all intersections between $\mathcal{E}$ and the x and y axes.

(b) *(5 points)* Convert $\mathcal{E}$ to the standard form for an ellipse

$$\frac{(x-x_0)^2}{a} + \frac{(y-y_0)^2}{b} = 1$$

(c) *(5 points)* Graph the ellipse $\mathcal{E}$.

(d) *(5 points)* Find a parametric equation for $\mathcal{E}$ using sine and cosine.

a) x-axis, $y=0$

$x^2-4x+4=1$

$x^2-4x+3=0$

$(x-3)(x-1)=0$

$x=1$ or 3

y-axis, $x=0$

$4(y-1)^2=1$

$4y^2-8y+3=0$

$(2y-1)(2y-3)=0$

$y=\frac{1}{2}$ or $\frac{3}{2}$

$(1,0), (3,0), (0,\frac{1}{2}), (0,\frac{3}{2})$

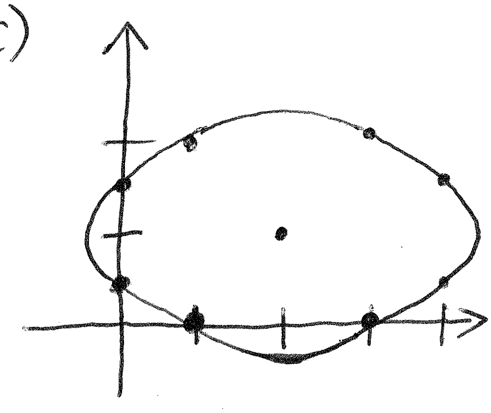
b) $x^2-4x+4(y-1)^2=1$

$x^2-4x+4+4(y-1)^2=5$

$(x-2)^2+4(y-1)^2=5$

$$\frac{(x-2)^2}{5} + \frac{(y-1)^2}{\left(\frac{5}{4}\right)} = 1$$

c)



d)

$x(t) = \sqrt{5}\cos t + 2$

$y(t) = \frac{\sqrt{5}}{2}\sin t + 1$