

Symmetric Surfaces from dynamics on triangles.

Description

Half-dilation surfaces are fun to build; you can snap together triangles like Magna-tiles®. I will describe a construction of half-dilation surfaces built from triangles produced by a $(\mathbb{Z}/2\mathbb{Z})(\mathbb{Z}/2\mathbb{Z})(\mathbb{Z}/2\mathbb{Z})$ action on homothety-equivalence classes of triangles in the plane. The advantage of this construction is that it produces surfaces with non-elementary Veech groups. Some of the surfaces that arise have infinite type, some others are already well-known: the Bouw-Möller lattice surfaces. This talk is about joint work with Seth Foster and Zhi Heng Liu.

Translation, Dilation, Affine and other Structures on Surfaces

Apr 7 – 11, 2025

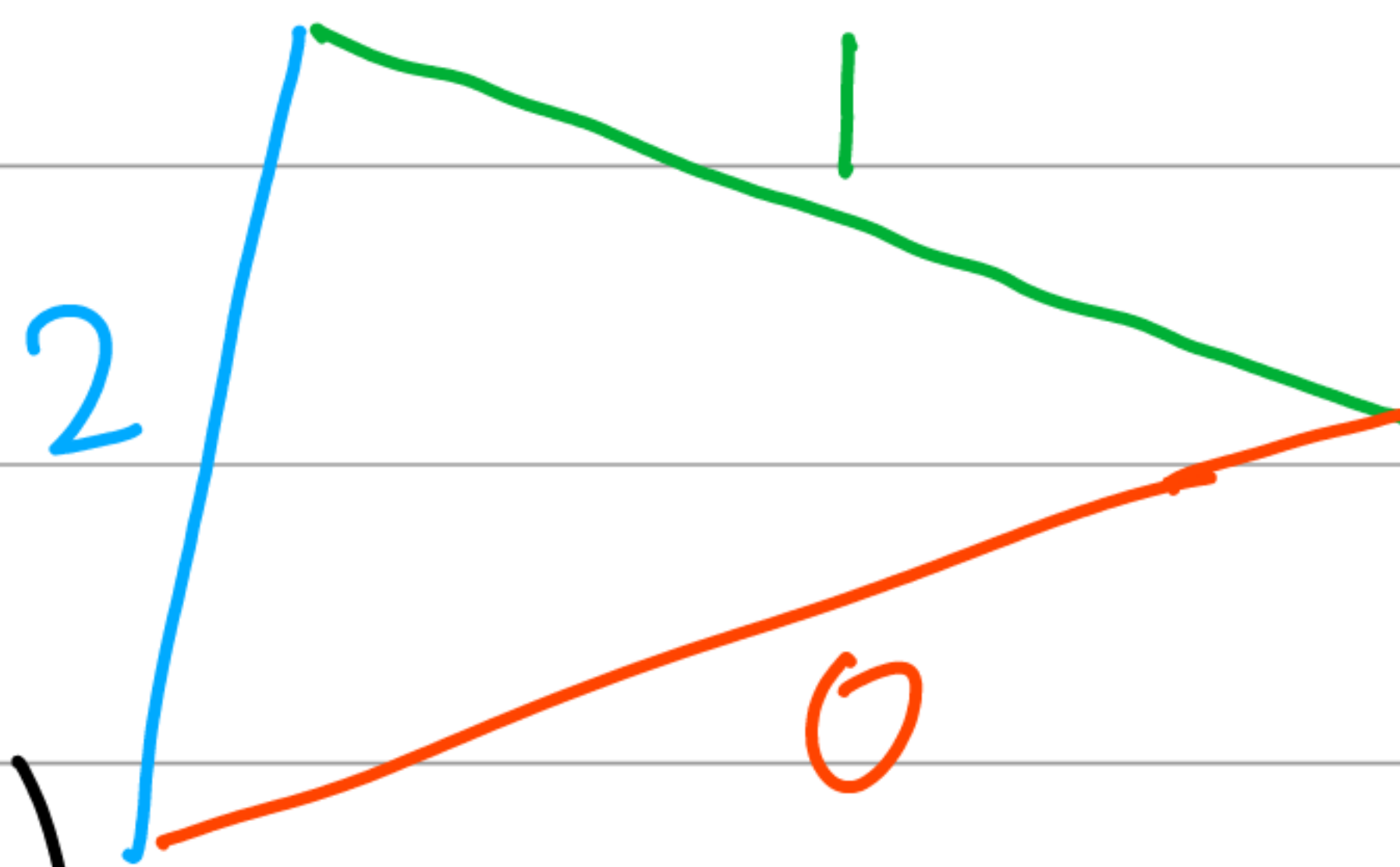
Institut de Mathématiques de Toulouse

Enter your sea

I. Triangles

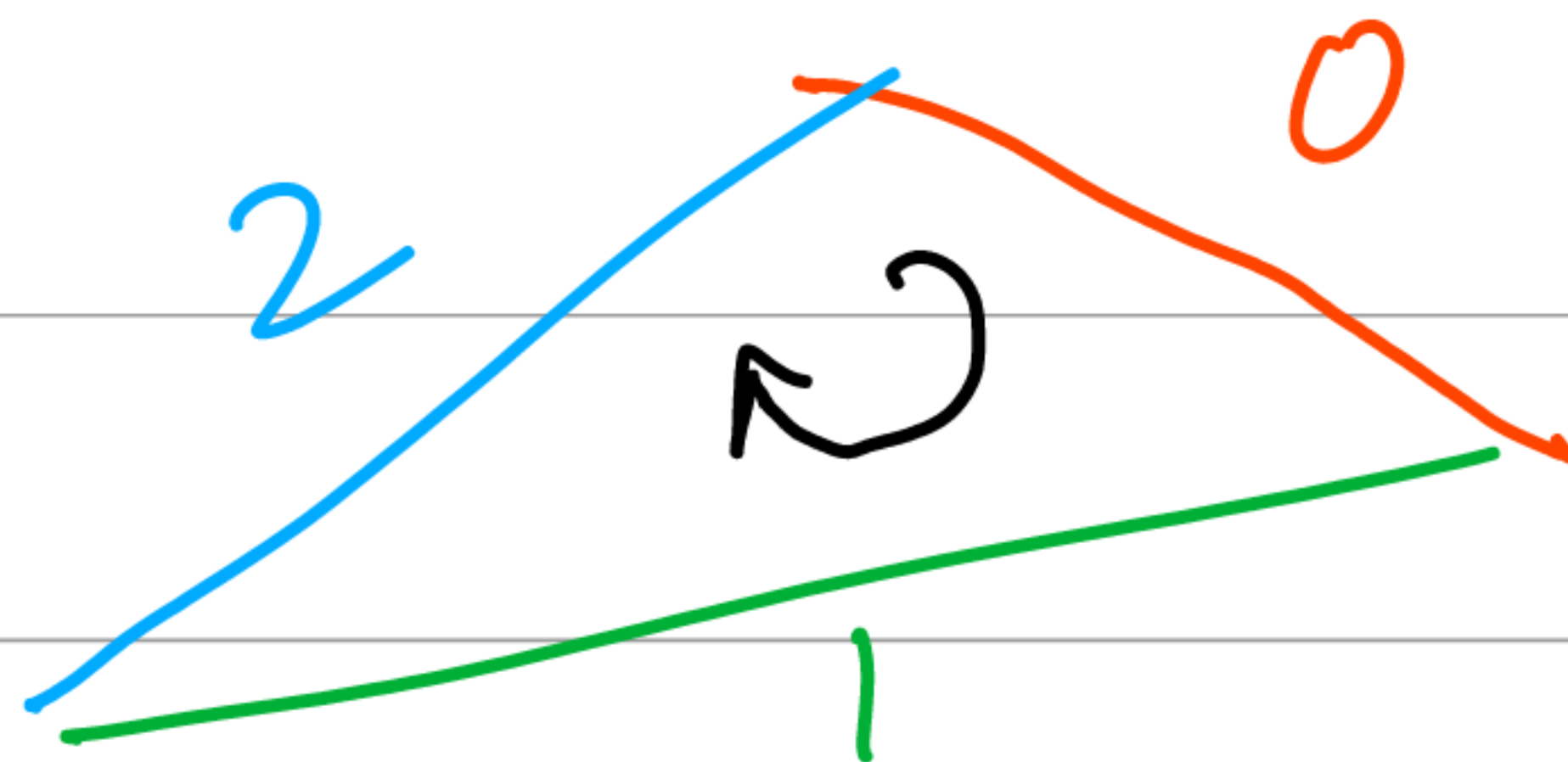
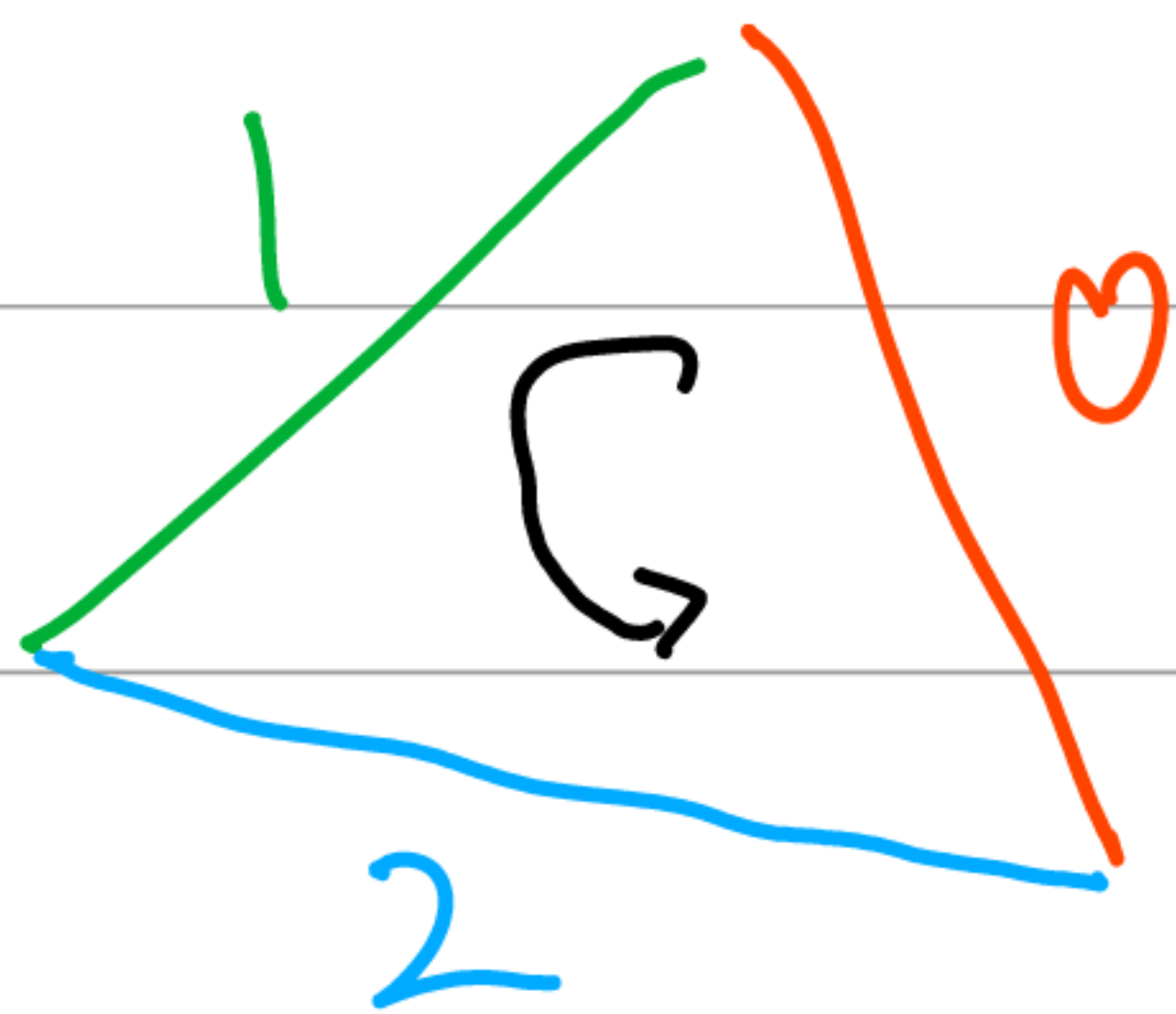
A marked triangle is a triangle in \mathbb{R}^2 with edges labeled by $\{0, 1, 2\}$.

Associated to a marked triangle is its triple of slopes $\vec{m} = (m_0, m_1, m_2)$ in $\hat{\mathbb{R}}^3$ where $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.



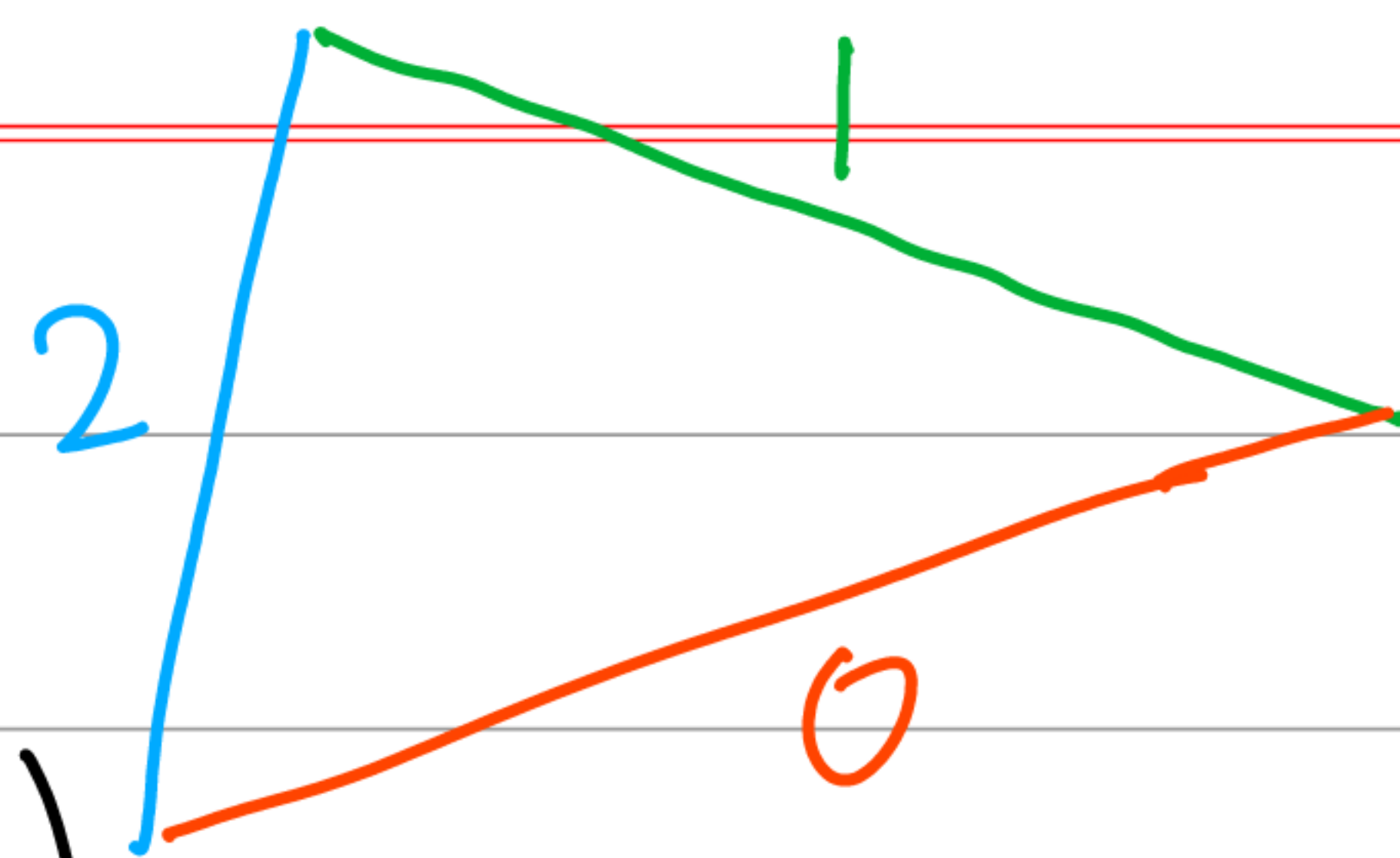
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positively oriented or negatively oriented
Depends on cyclic order of edge labels.

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The homothety or half-dilation group is

$$HD = \left\{ z \mapsto az + b : a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{C} \right\} \subset \text{Aff}(\mathbb{C}).$$

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Furthermore, a triangle is positively oriented iff its slope triple is in decreasing cyclic order.

II. Quadrilaterals.

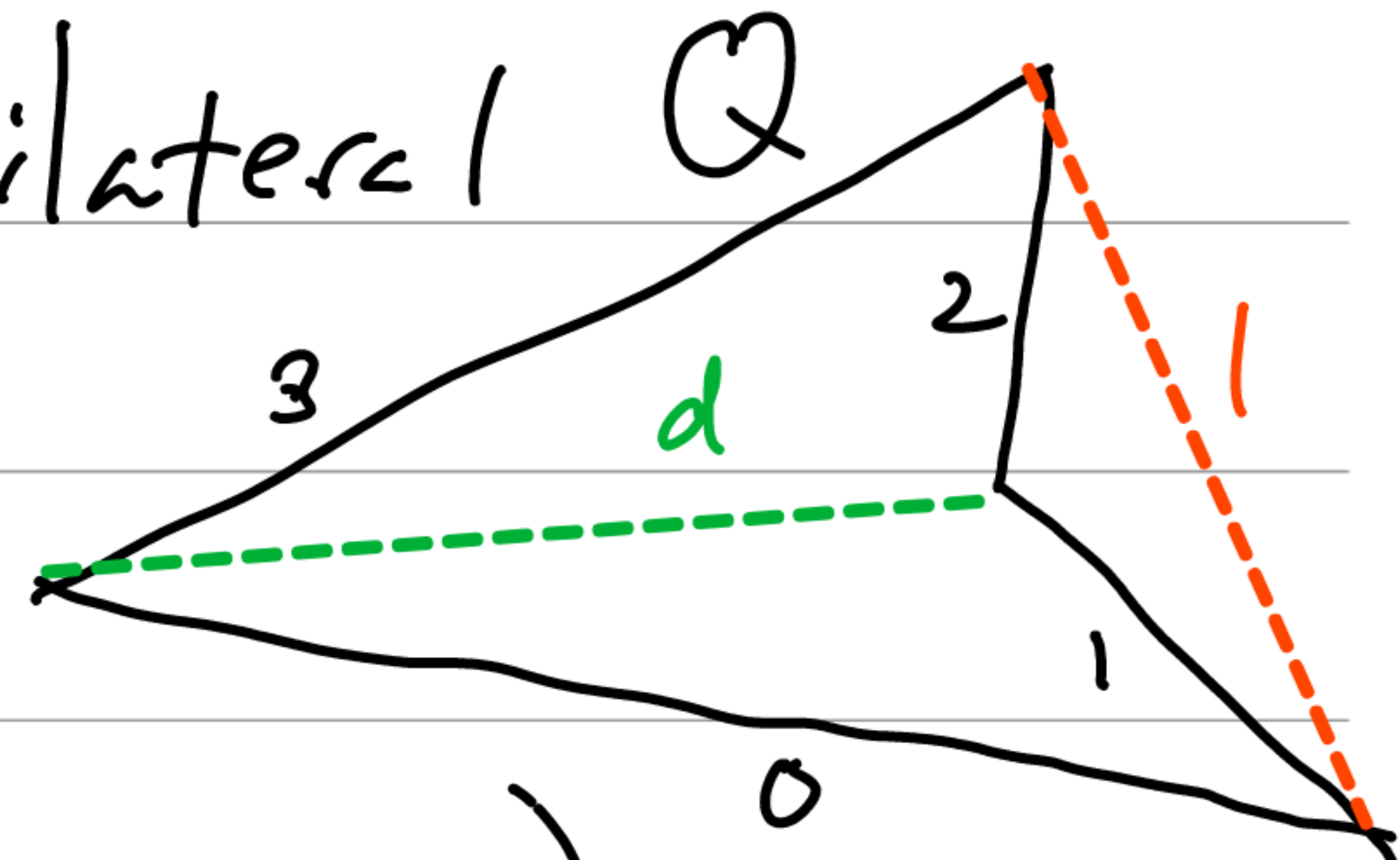
Setup: $GL(2, \mathbb{R}) / \underbrace{HD \cap GL(2, \mathbb{R})}_{(\mathbb{R} \setminus \{0\}, \times)} = PGL(2, \mathbb{R})$
 $= \text{Möb}(\mathbb{R})$
 $= \text{Isom}(\mathbb{H})$

where $\text{Möb}(\mathbb{R}) = \left\{ z \mapsto \frac{az+b}{cz+d} : \begin{array}{l} a, b, c, d \in \mathbb{R}, \\ ad-bc \neq 0 \end{array} \right\}$

is the isometry group of the hyperbolic plane $\mathbb{H} = (\mathbb{C} \setminus \mathbb{R}) / \text{complex conjugation}$.

(We identify the upper and lower half planes via $z \mapsto \bar{z}$.)

Consider a marked quadrilateral Q with diagonals D and d .



Quad. Thm 1 If $(m_0, m_1, m_2, m_3; m_D, m_d)$

is the slope data of any quadrilateral Q that is non-degenerate in the sense that $m_k \neq m_{k+1} \pmod{4} \quad \forall k \in \{0, 1, 2, 3\}$,

then \exists an involution $L_Q = \frac{az+b}{cz-a} \in \text{Möb}(\mathbb{R})$

s.t. $m_0 \xleftrightarrow{L_Q} m_2, \quad m_1 \xleftrightarrow{L_Q} m_3, \quad m_D \xleftrightarrow{L_Q} m_d.$

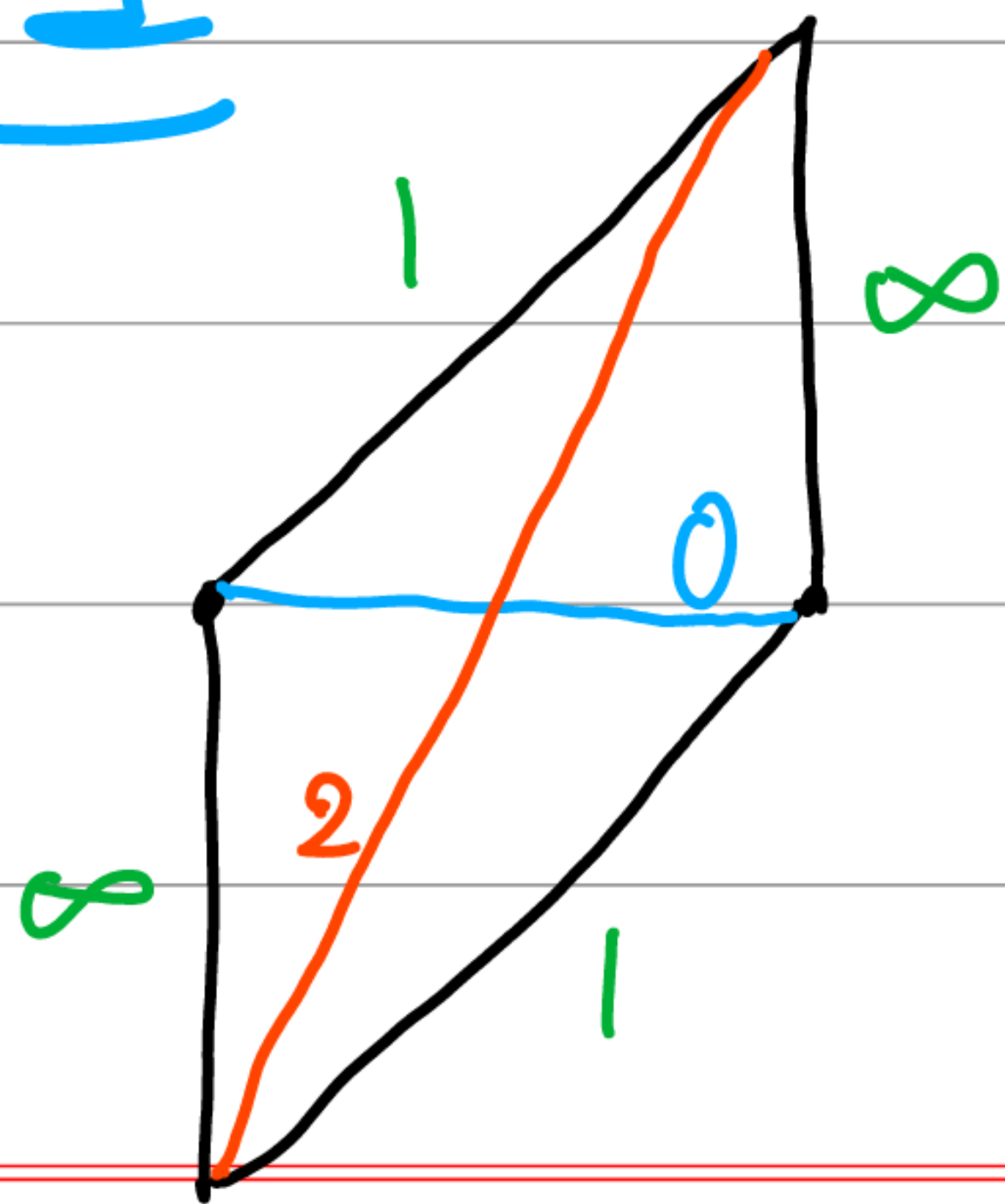
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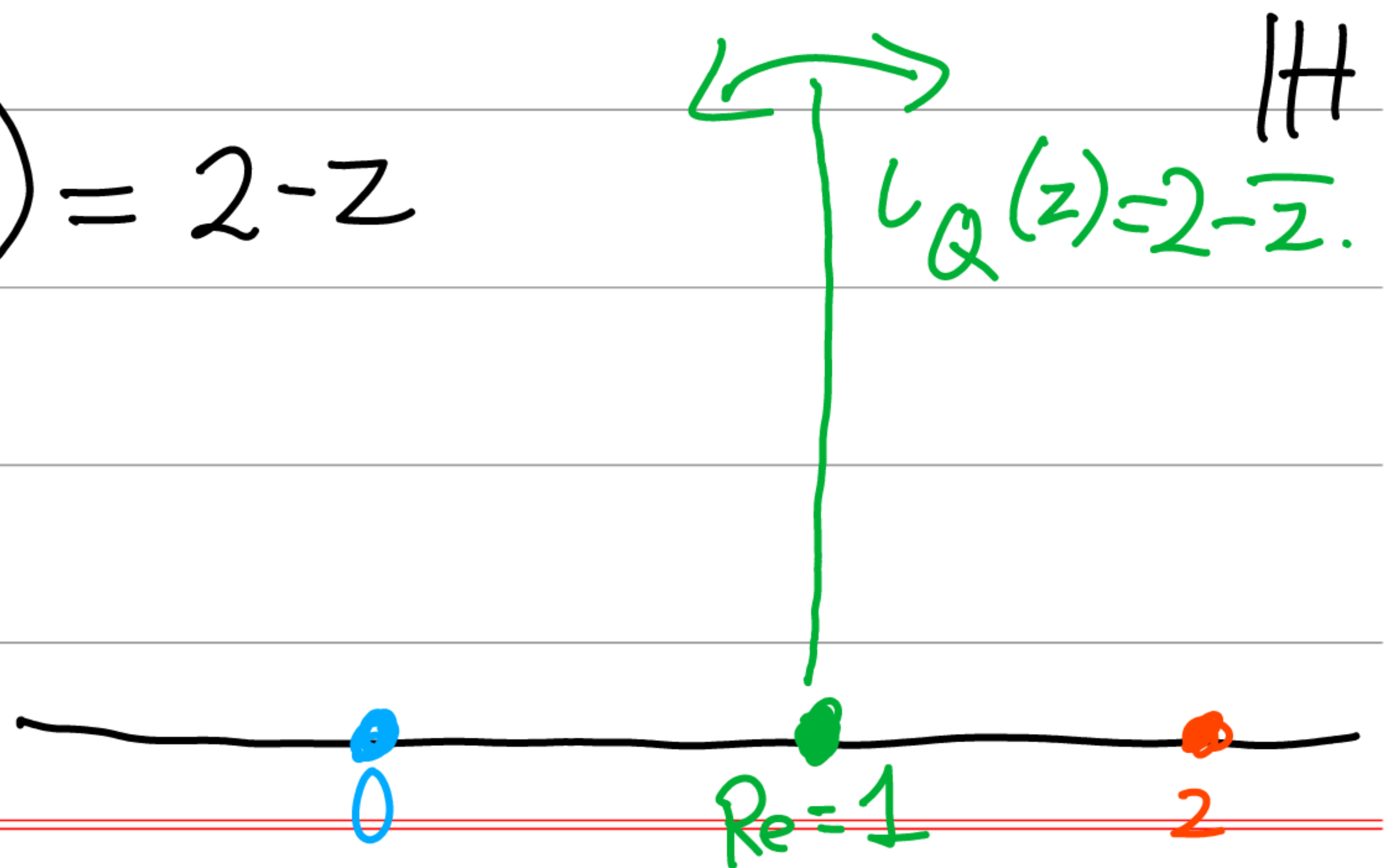
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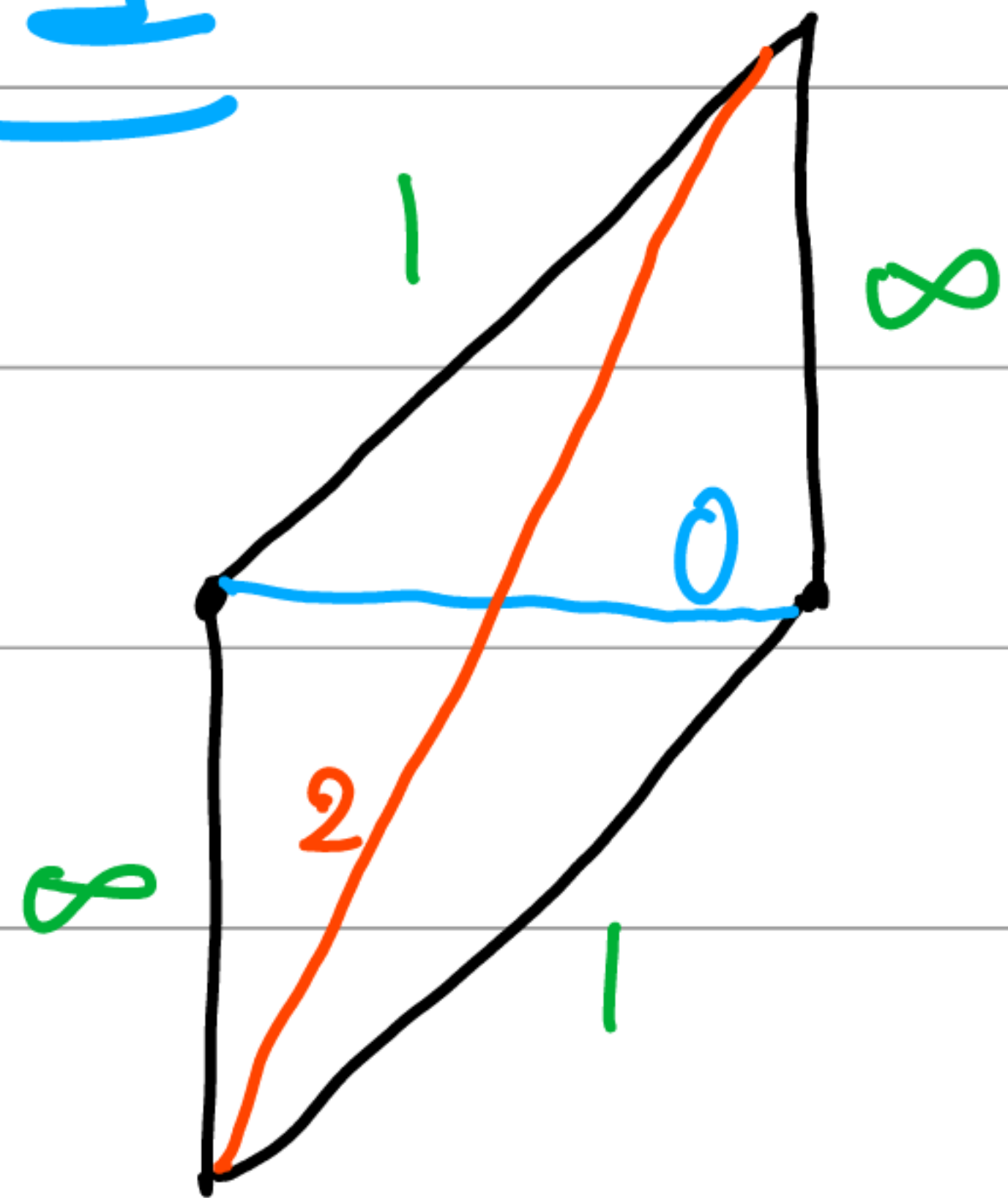
Ex 1:



$$L_Q(z) = 2 - z$$



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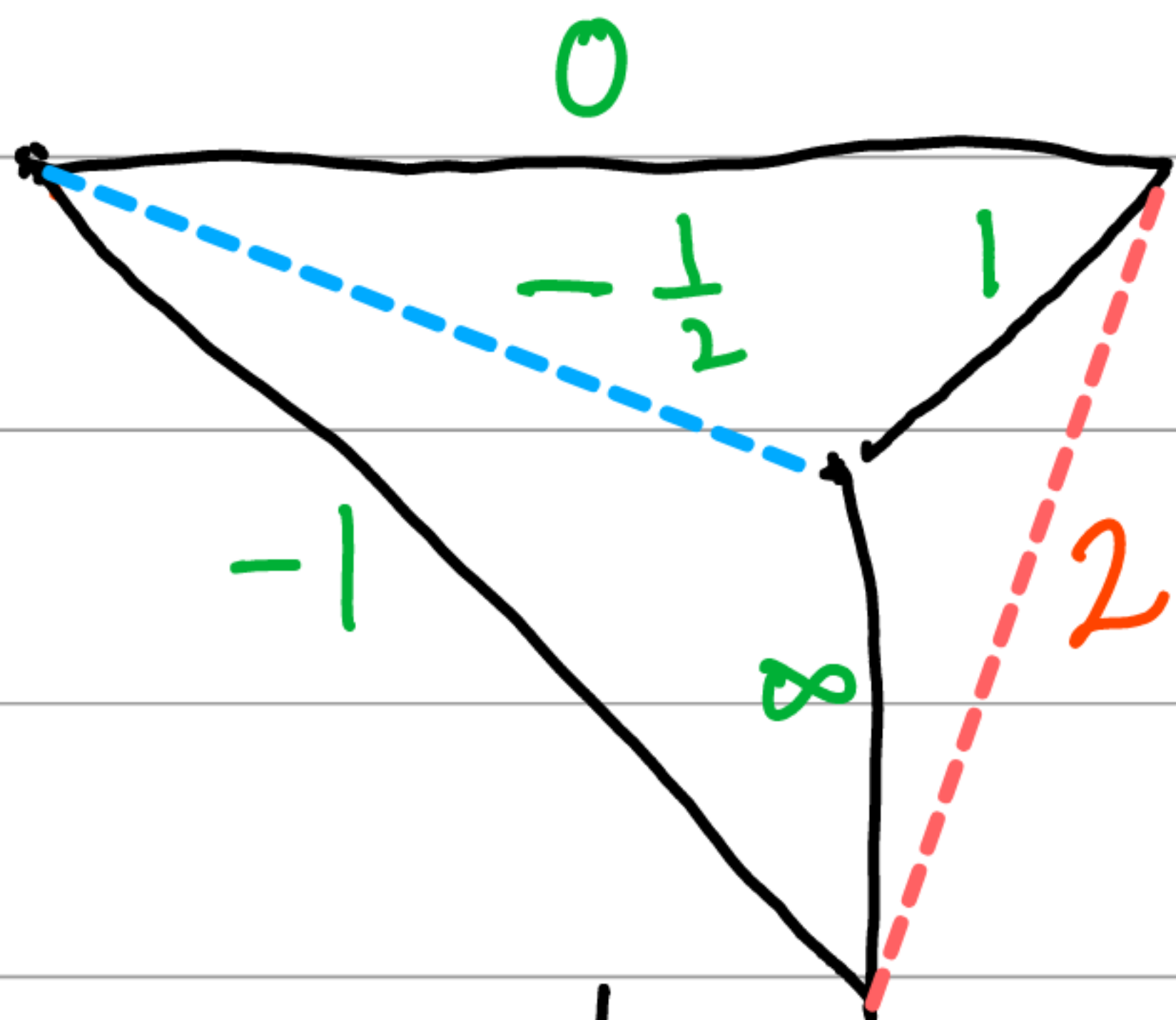


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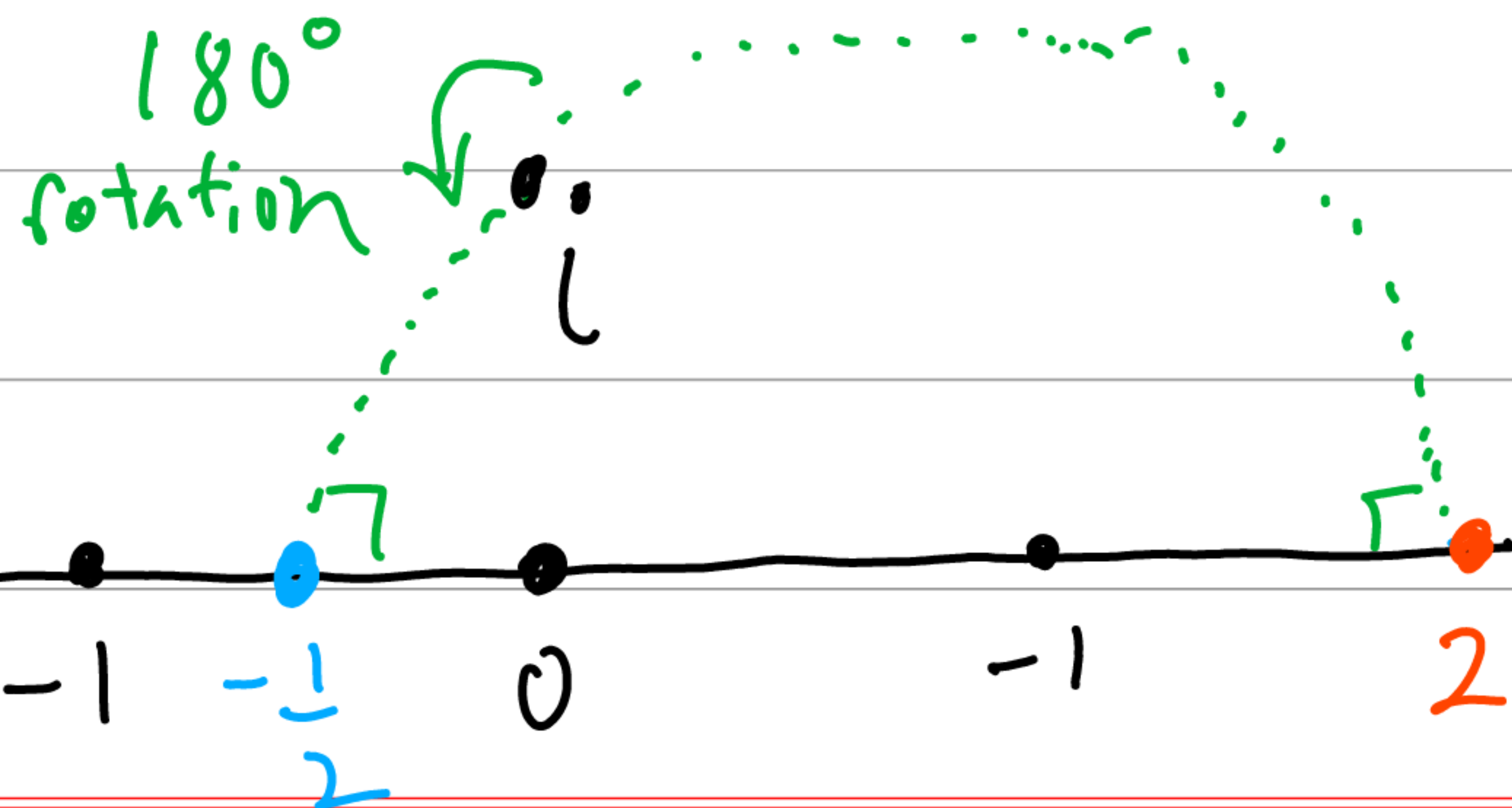
$$l_Q(z) = 2 - \bar{z}$$



Ex 2



$$l_Q(z) = \frac{1}{z}$$



Quad Thm 2 A non-degenerate

quadrilateral Q is convex

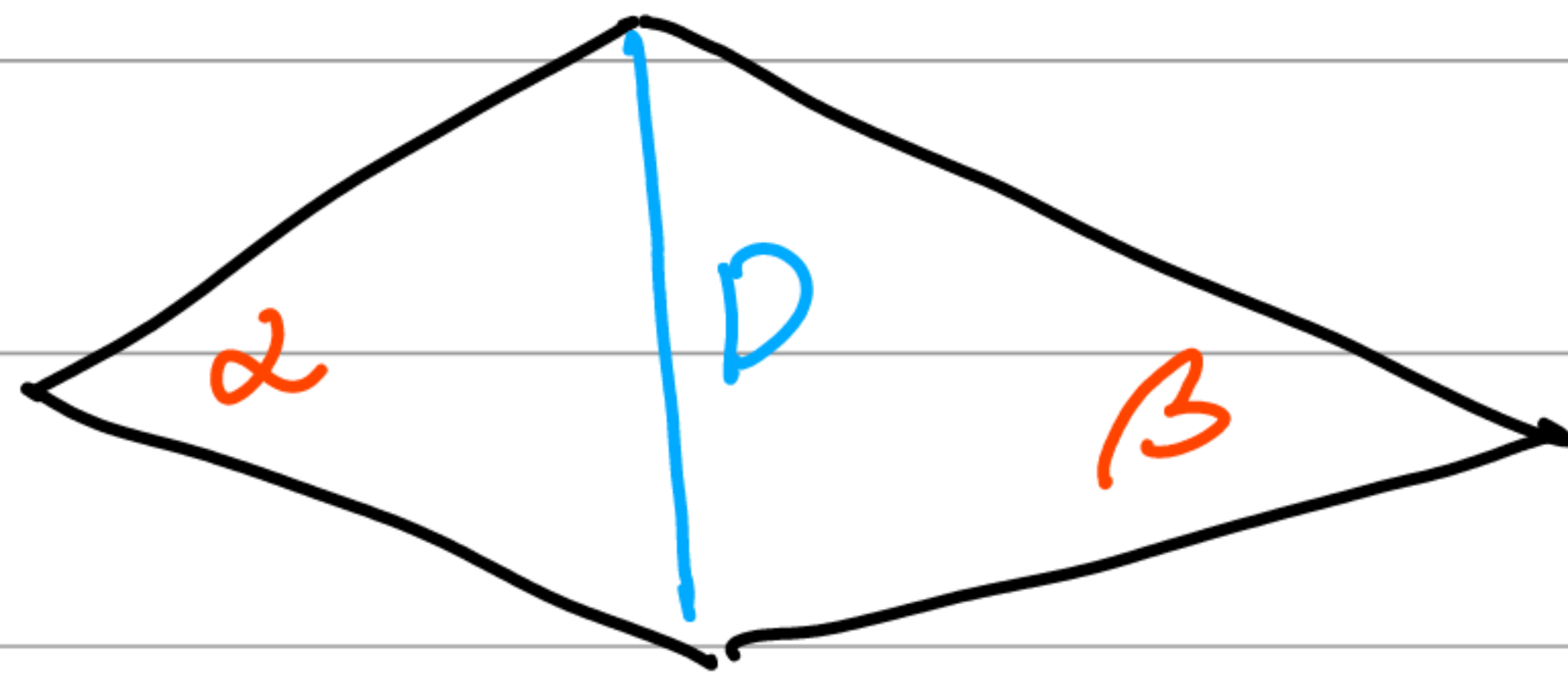
iff ℓ_Q is orientation reversing

on both $\hat{\mathbb{R}}$ and H .

Now assume Q is convex. A diagonal

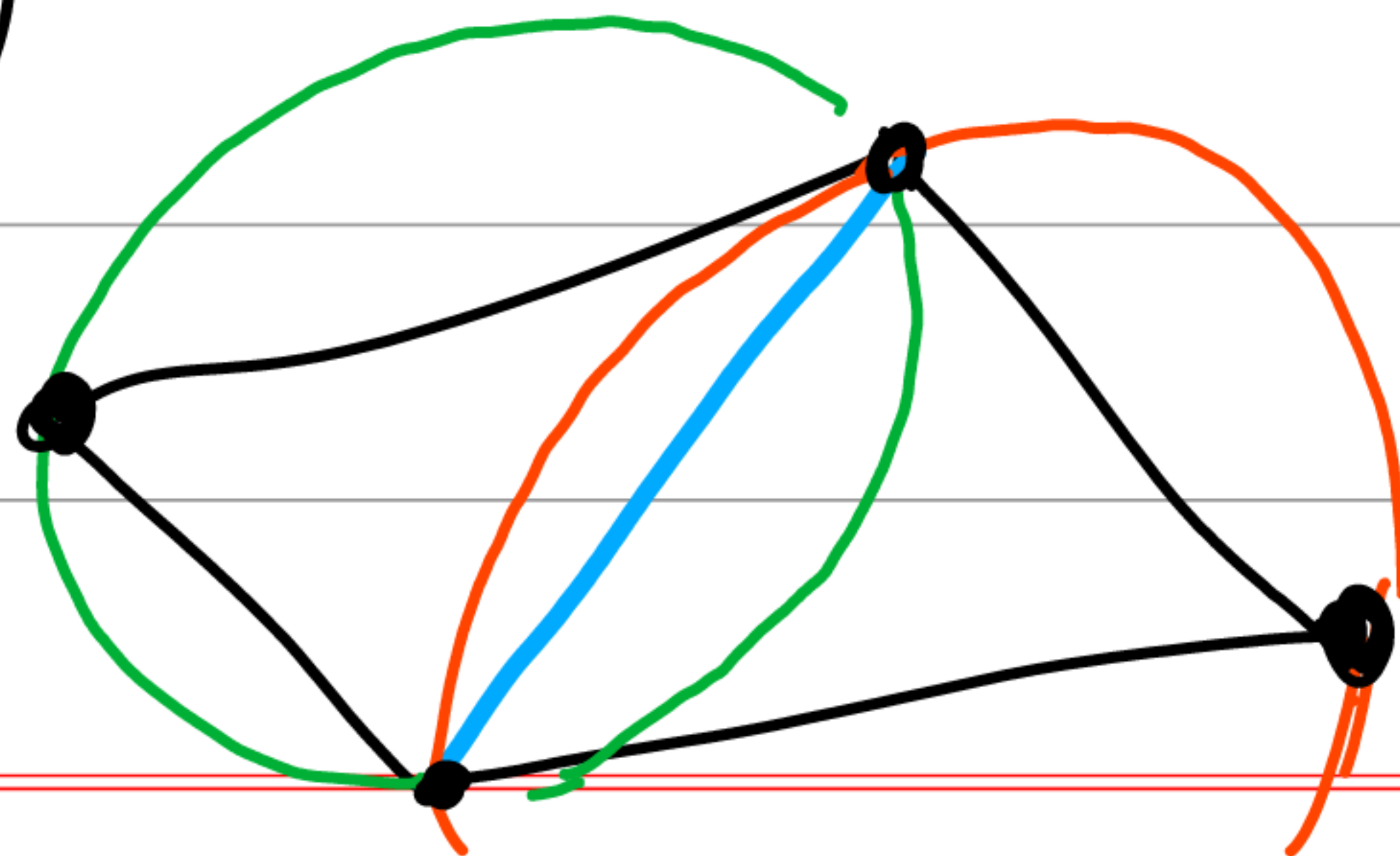
D is Delannay if opposite angles

sum to $\leq \pi$.



$$\alpha + \beta \leq \pi$$

(Equivalently, circles inscribing the resulting triangles do not contain the remaining vertex of Q in their interiors.)



Quad Thm 3: Let Q be strictly convex.

Let $\gamma = \text{Fix}(\iota_Q) \subset \overline{\mathbb{H}}$ which is a geodesic.

Then D is Delaunay iff either

$i \in \gamma$ or i lies on the same side of γ as m_D .

case when Q is inscribed in a circle.

Quad Thm 3: Let Q be strictly convex.

Let $\gamma = \text{Fix}(\iota_Q) \subset \overline{\mathbb{H}}$ which is a geodesic.

Let $\gamma \in \text{Möb}(\mathbb{R}) = \text{PSL}(2, \mathbb{R})$. Then

$\gamma(D)$ is diagonal in $\tau(Q)$ iff

$\tau^{-1}(i) \in \gamma$ or if $\tau^{-1}(i)$ lies on the

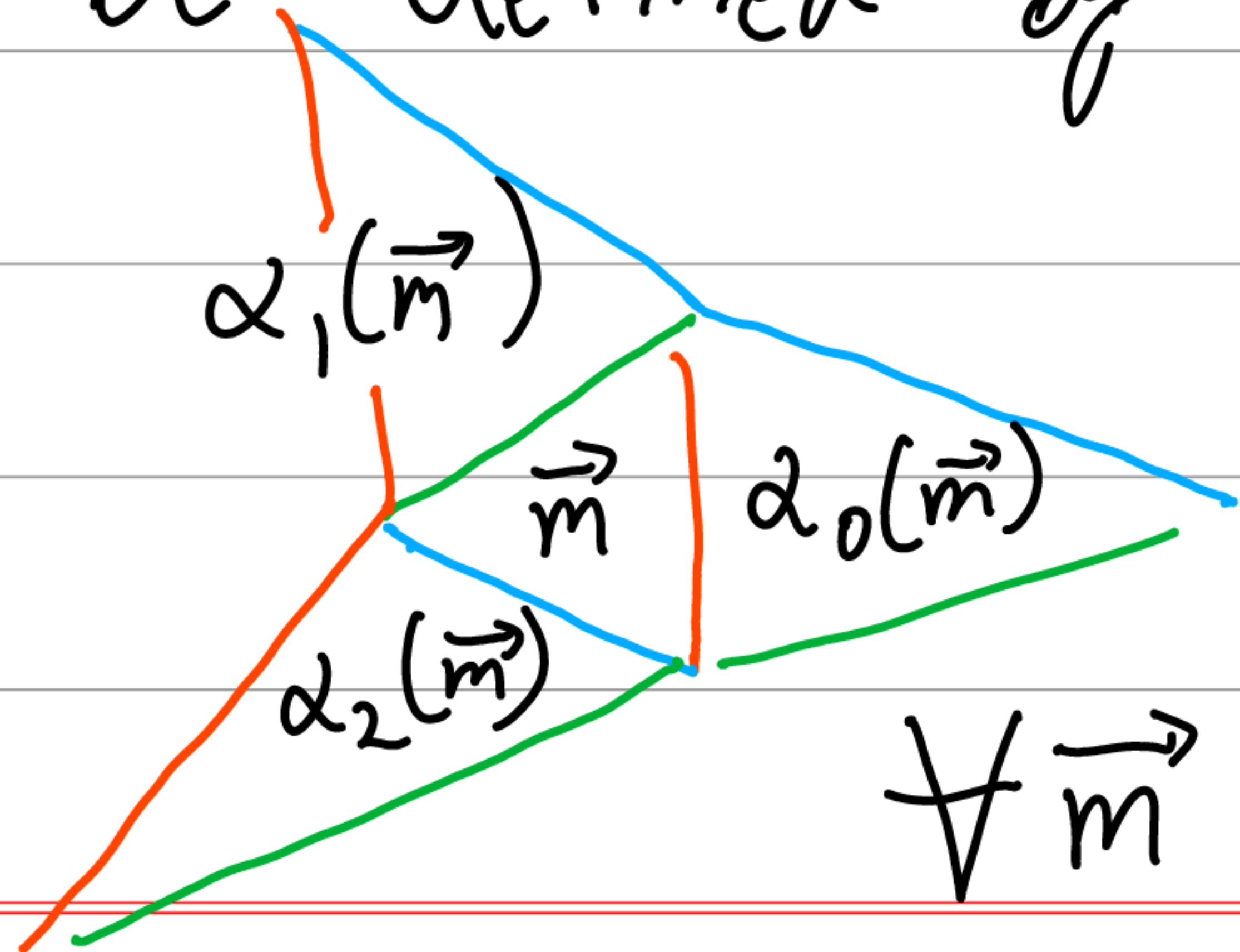
same side of γ as m_D .

III. Constructing Surfaces

Choose a triple of orientation-reversing Möbius involutions $\vec{c} = (c_0, c_1, c_2)$.

For $i \in \{0, 1, 2\}$, let $\alpha_i: \hat{\mathbb{R}}^3 \rightarrow \hat{\mathbb{R}}^3$

be defined by $\alpha_i(\vec{m})_j = \begin{cases} c_i(m_j) & \text{if } i \neq j \\ m_j & \text{if } i = j. \end{cases}$



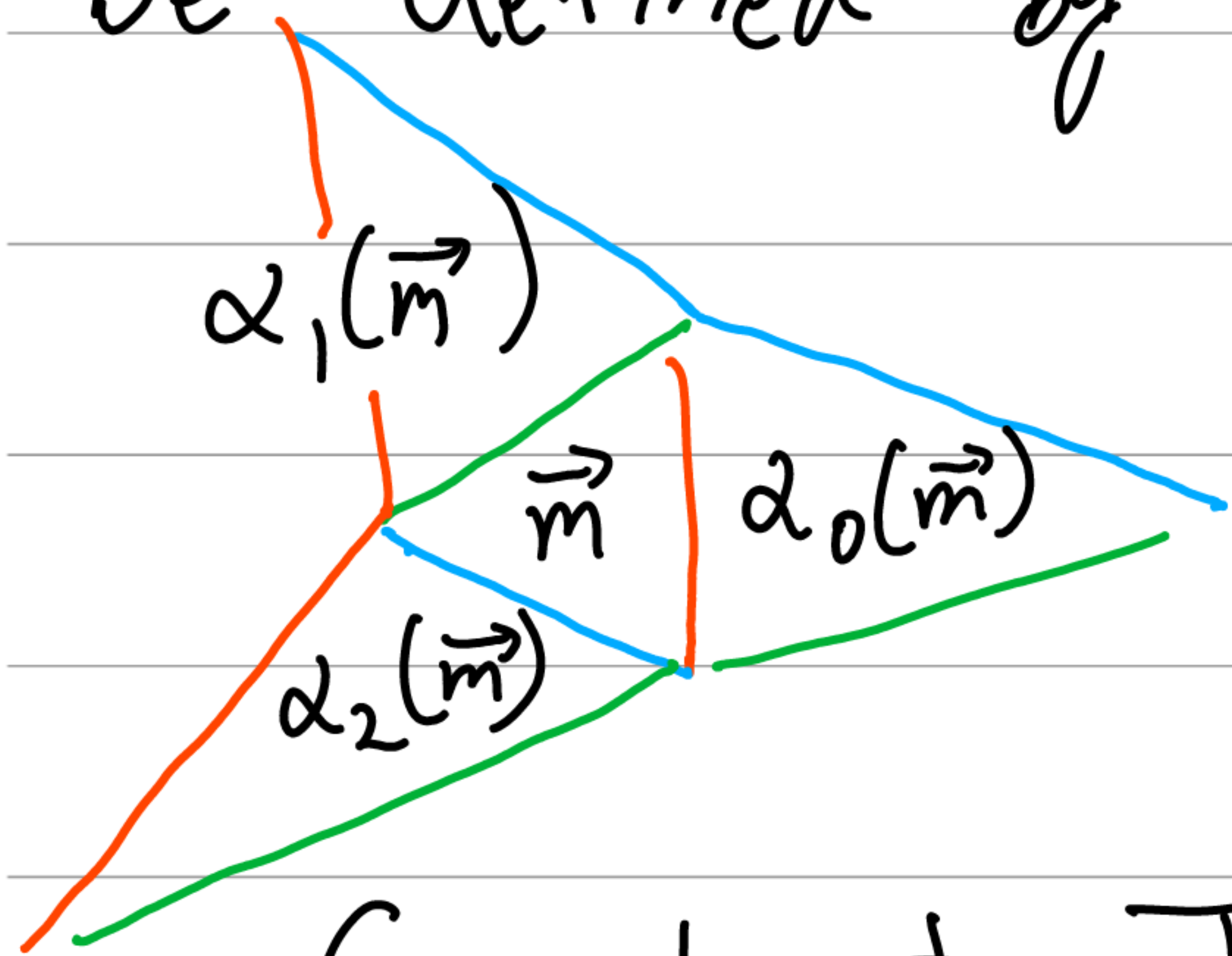
$$G = \langle \alpha_0, \alpha_1, \alpha_2 \rangle,$$
$$\mathcal{O} \subset \hat{\mathbb{R}}^3 \text{ a } G\text{-orbit.}$$

$\forall \vec{m} \forall i$, glue $T_{\vec{m}}$ to T

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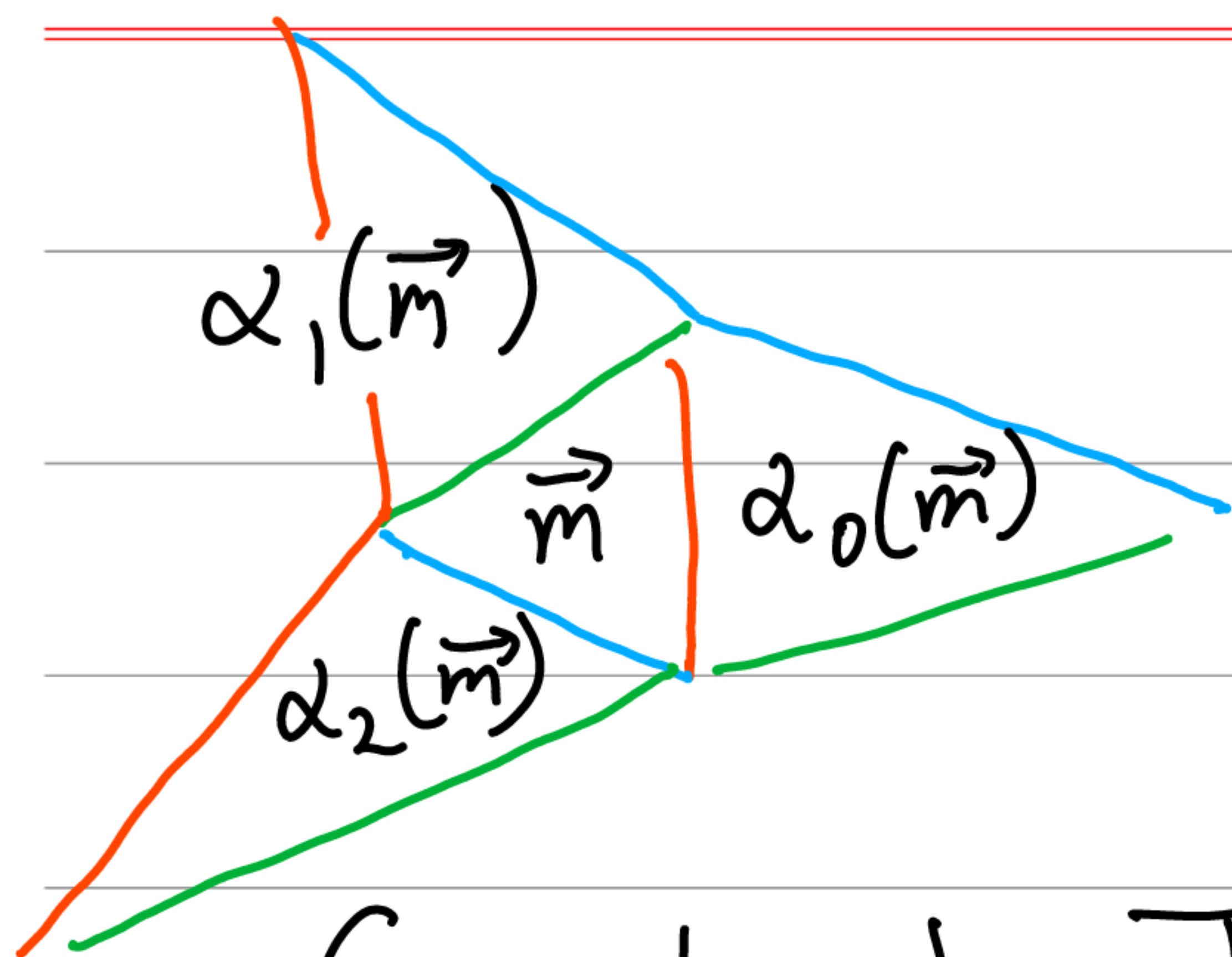


$$G = \langle \alpha_0, \alpha_1, \alpha_2 \rangle.$$

$\mathcal{O} \subset \widehat{\mathbb{R}^3}$ a G -orbit.

Construct $T_{\vec{m}} \forall \vec{m} \in \mathcal{O}$.

Glue edge i of $T_{\vec{m}}$ to edge i of $T_{\alpha_i(\vec{m})}$.



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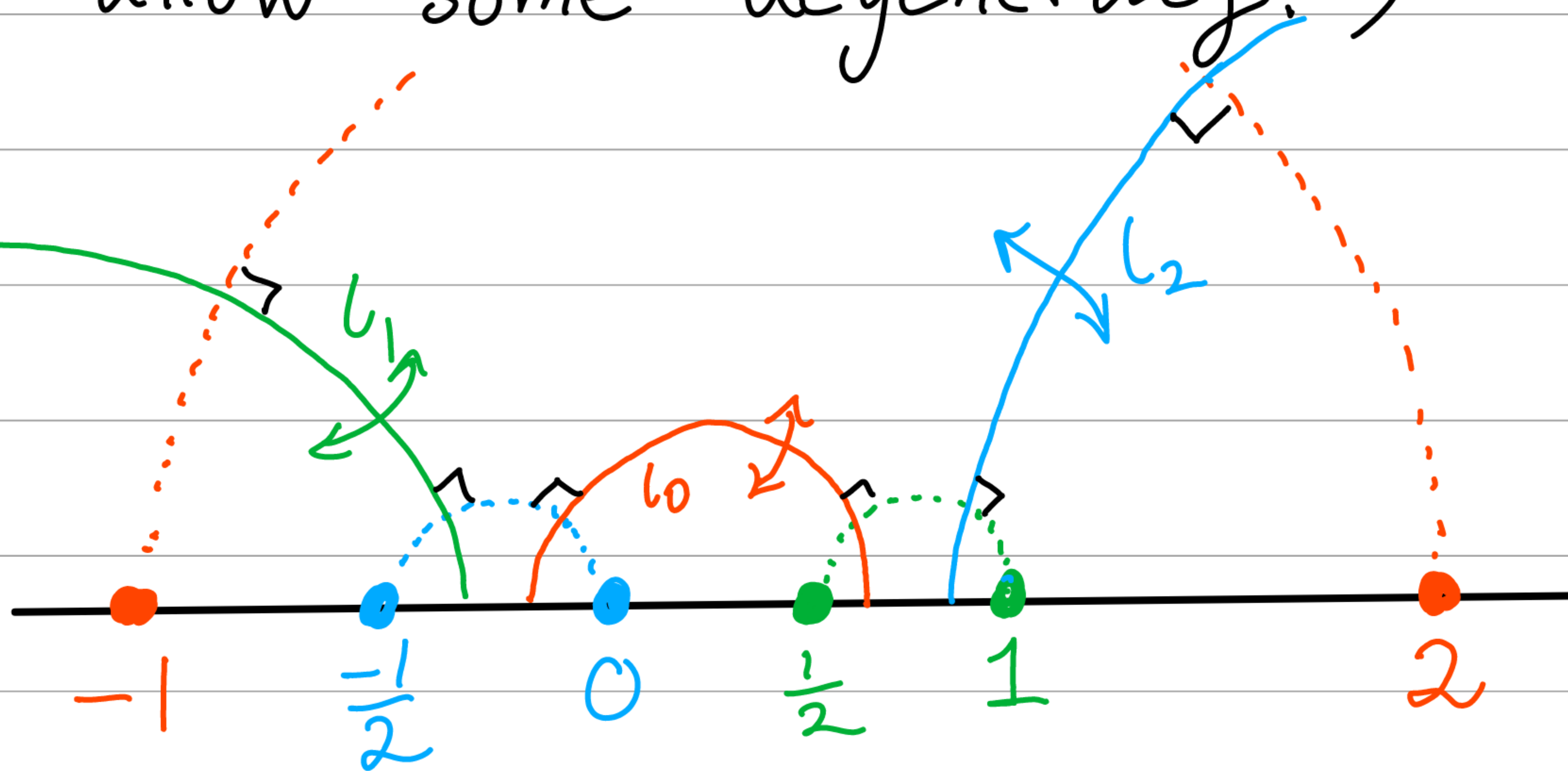
Glue edge i of $T_{\vec{m}}$ to edge i of $T_{\alpha_i(\vec{m})}$.

Def We call $S = S(\vec{t}, \mathcal{O})$ successful if all triangles produced have the same orientation.
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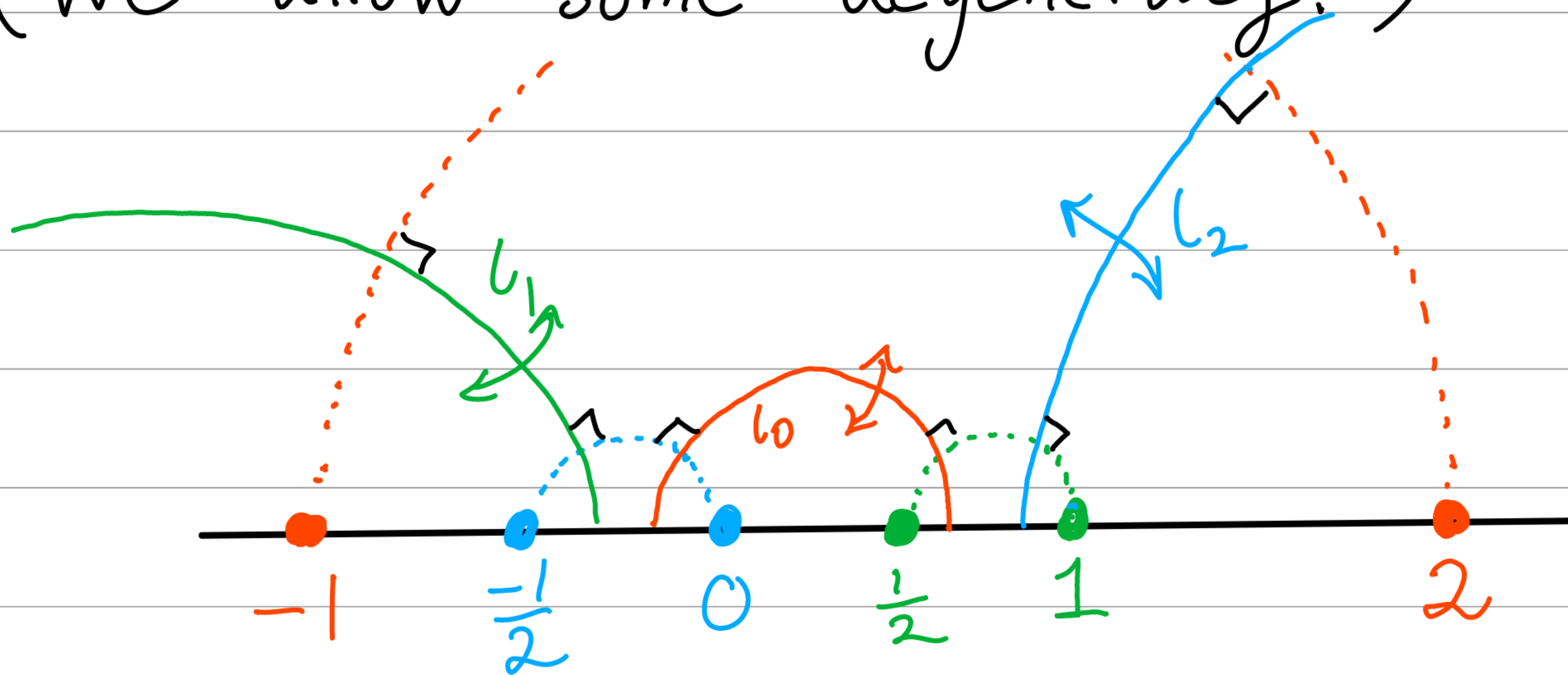
Ex

Red: 0
Green: 1
Blue: 2



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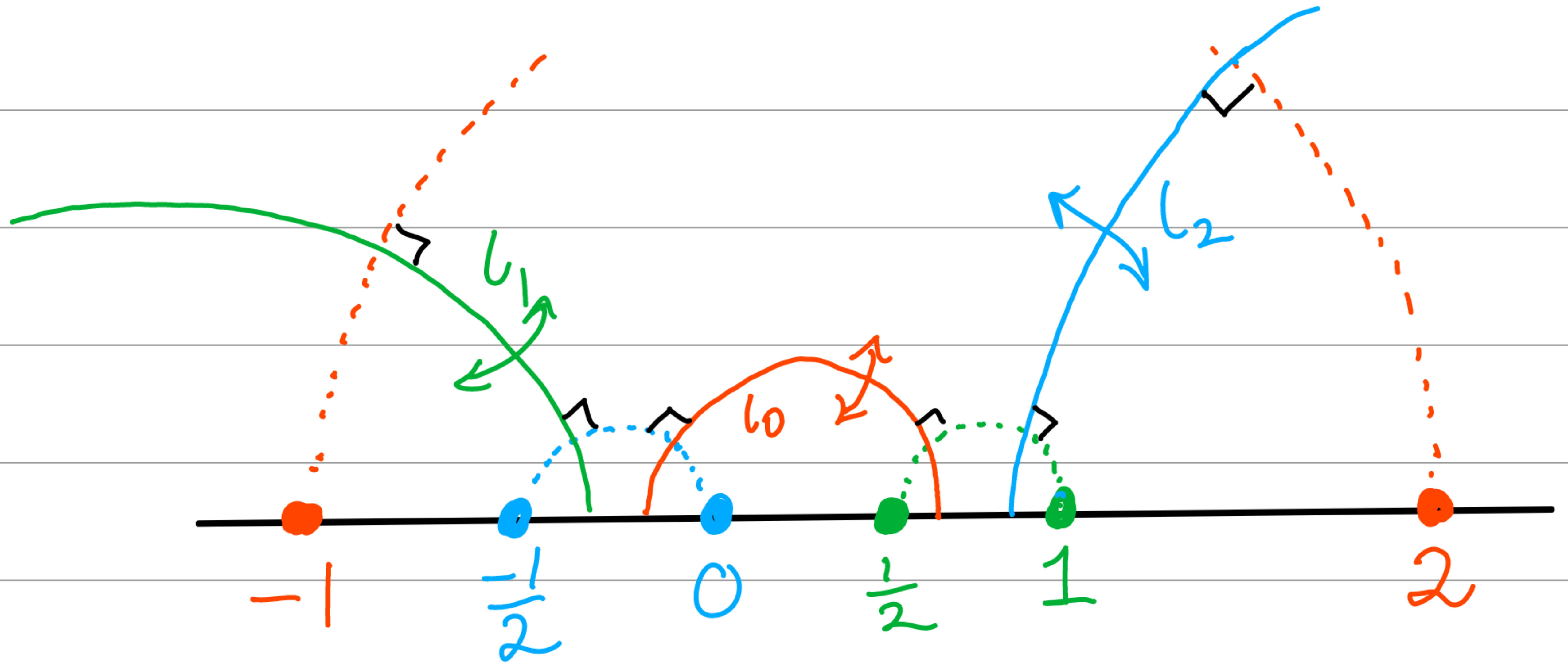
This construction will be successful because all triples (red, green, blue) decrease.

Ex

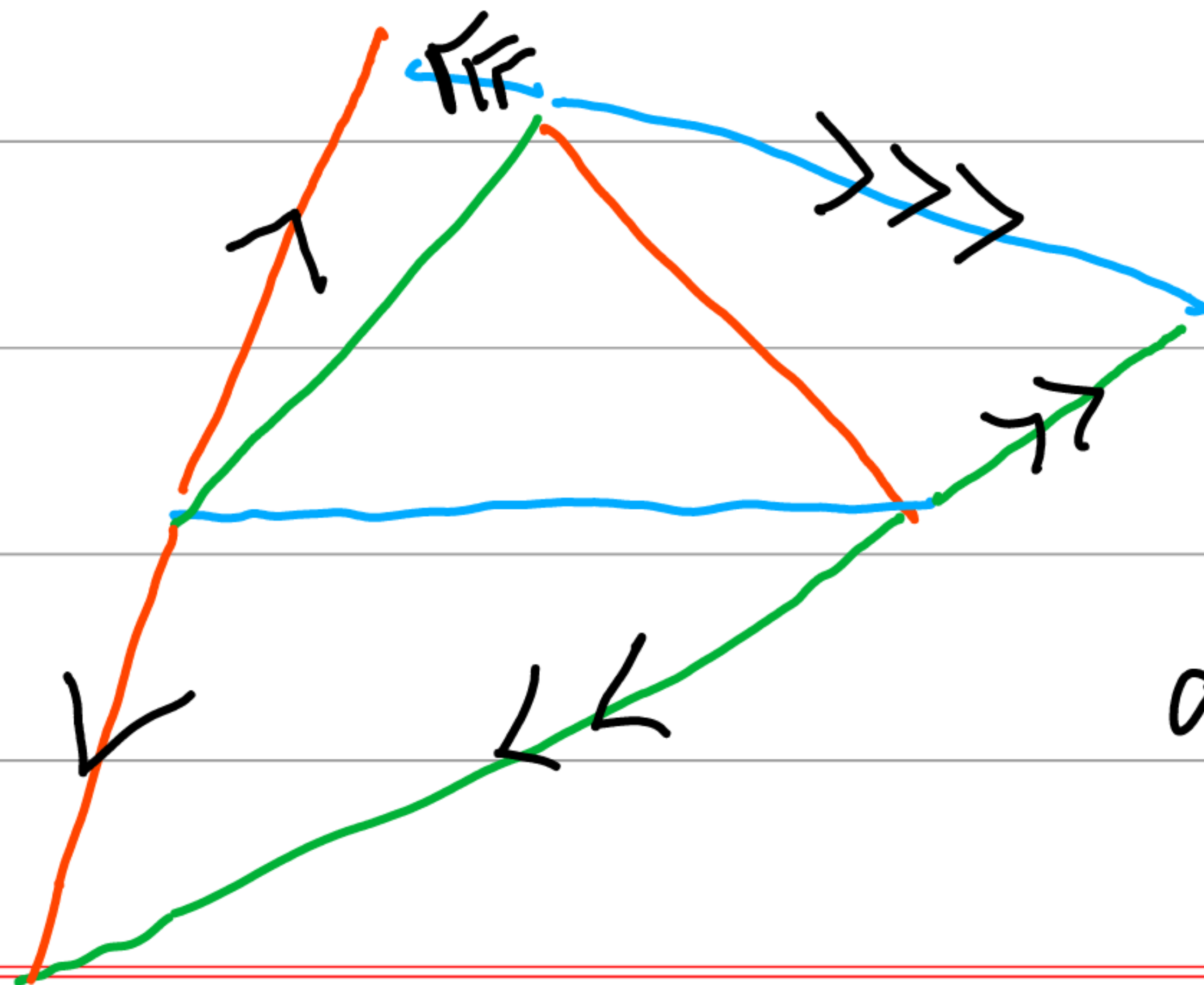
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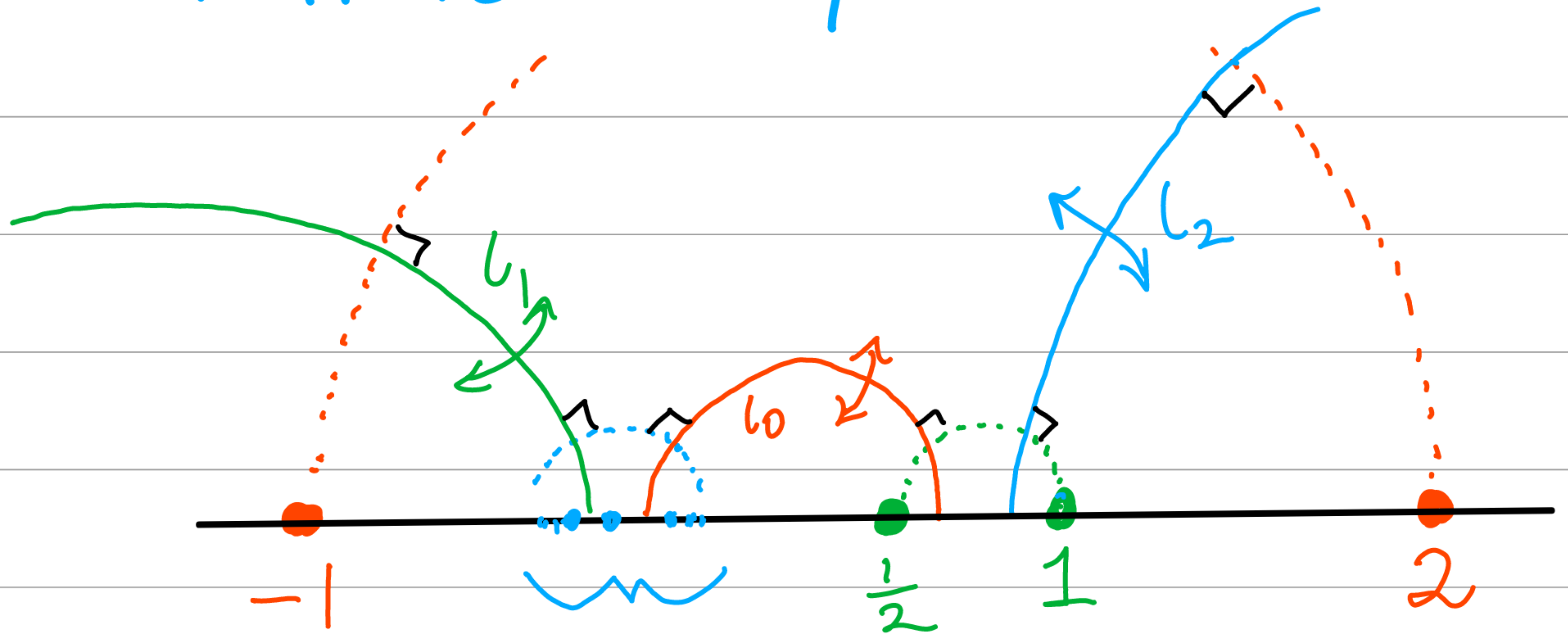
$$\mathcal{O} = G(-1, 0, 1). \quad S = S(\vec{t}, \mathcal{O}):$$



Studied by
Taro Shima
arXiv:2205.00100

Related infinite example:

Red: 0
Green: 1
Blue: 2



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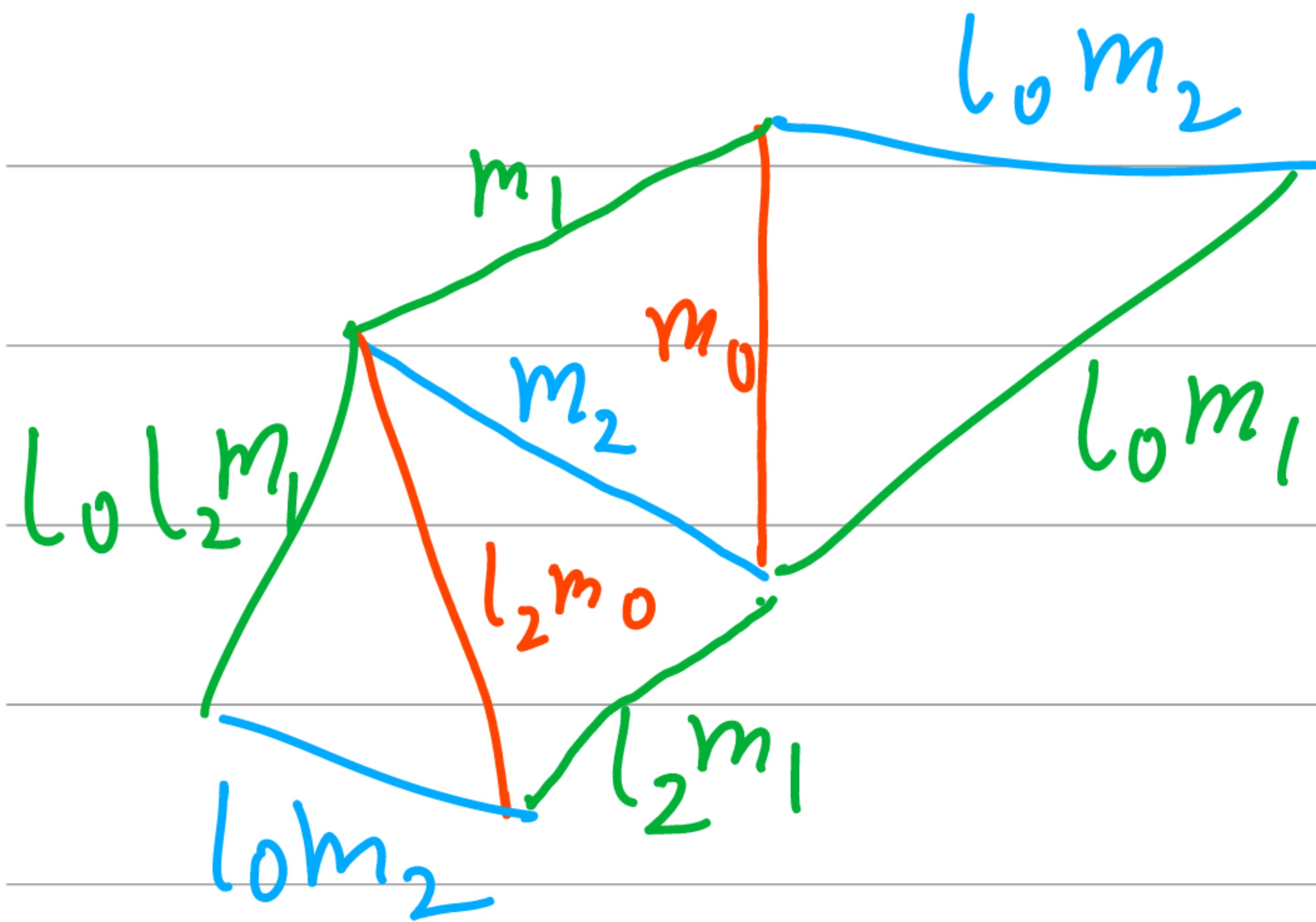
Thm The Veech group of a successfully constructed $S = S(\vec{t}, \theta)$ contains a subgroup of $\langle L_0, L_1, L_2 \rangle$ of at most index 8. This is the full Veech group up to index at most 6.

Thm If \mathcal{O} is finite, the orbit $\text{PSL}(2, \mathbb{R}) \cdot S$ is closed in the space of triangulable half-dilation surfaces.

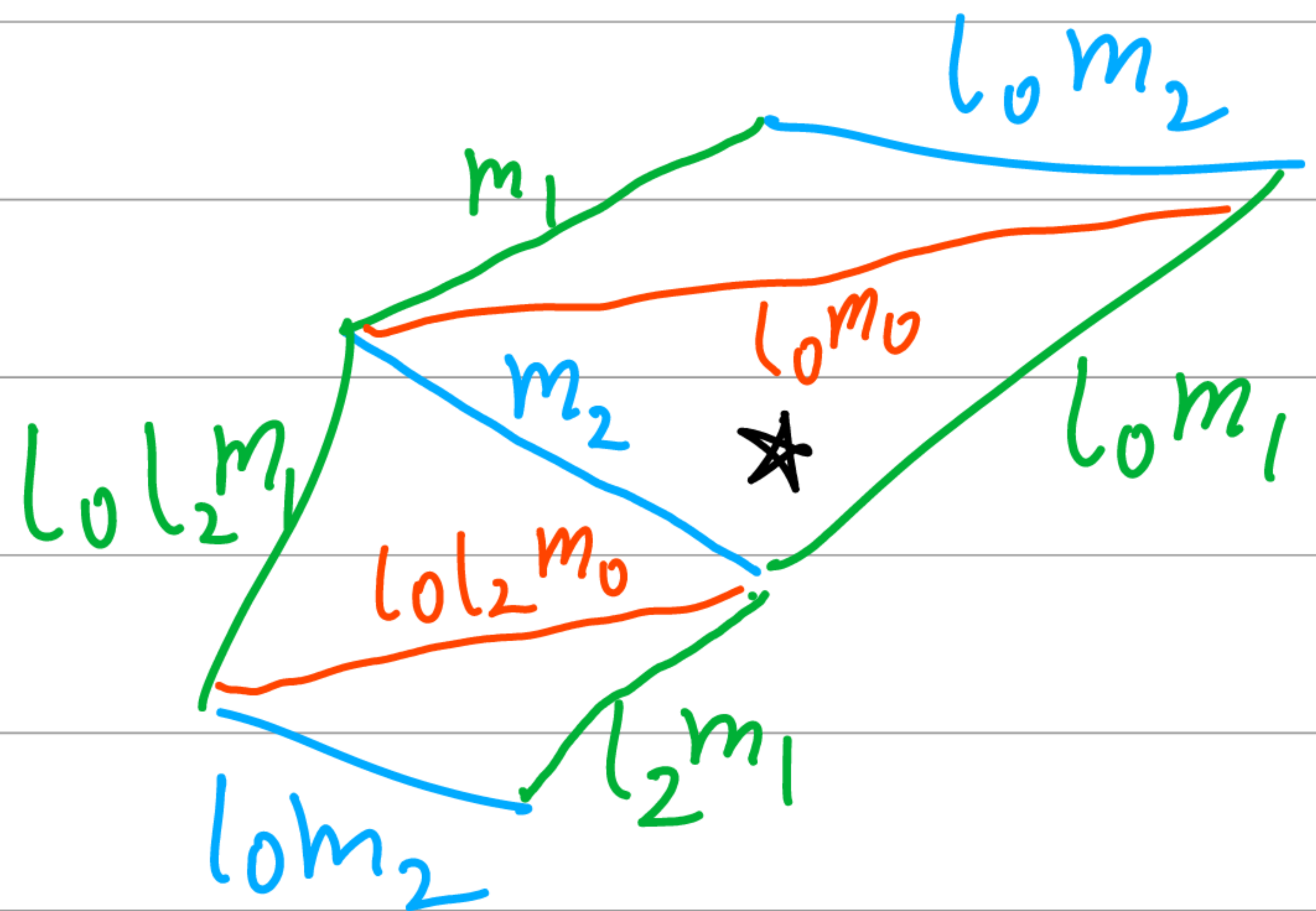
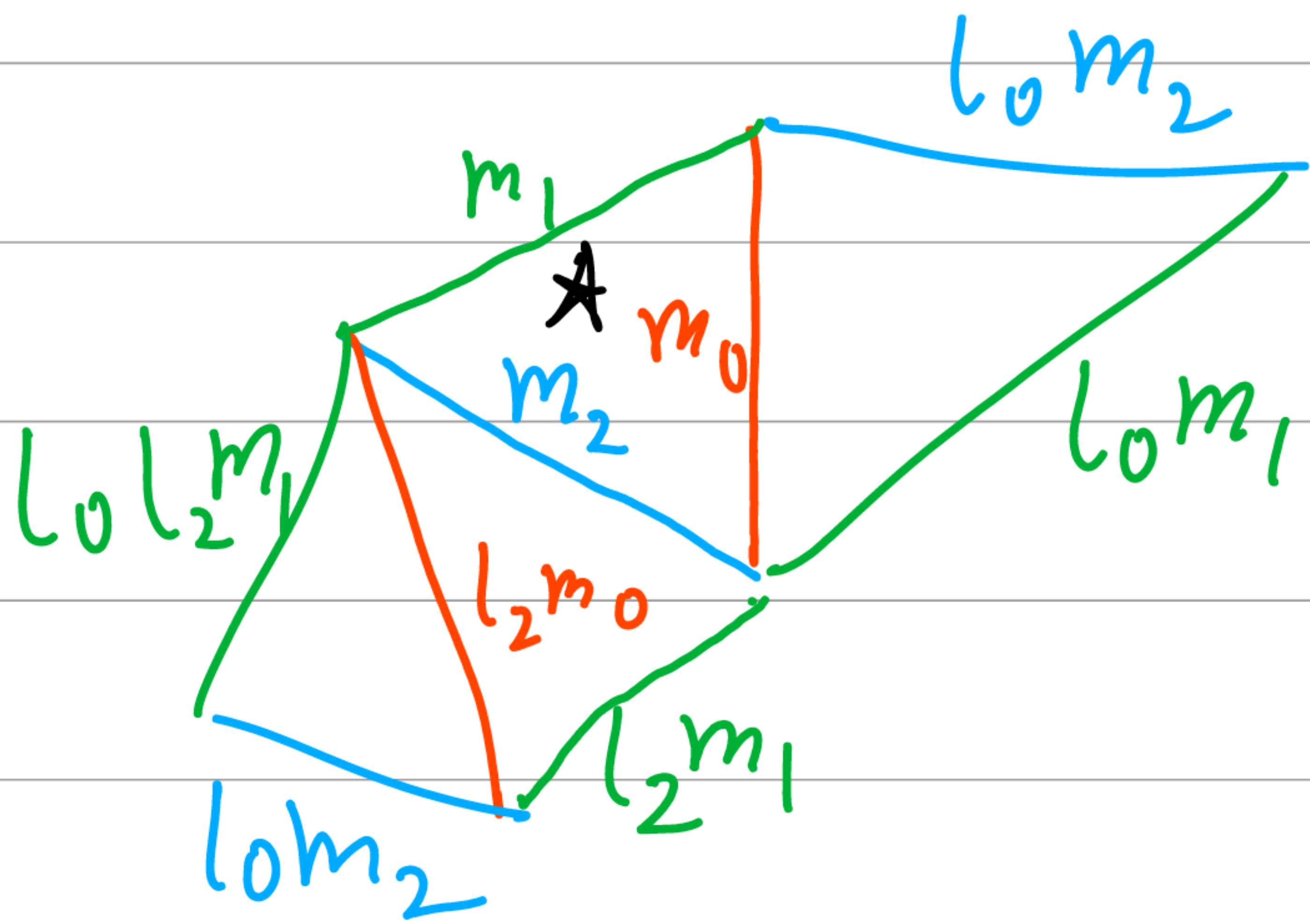
Thm The Veech group of a successfully constructed $S = S(\vec{l}, \theta)$ contains a subgroup of $\langle l_0, l_1, l_2 \rangle$ of at most index 8.

Lemma If we flip all edges with index 0, we go from $S(l_0, l_1, l_2, \theta)$ to $S(l_0, l_0 l_1 l_0, l_0 l_2 l_0, (l_0, I, l_0) \cdot \theta)$.

Lemma If we flip all edges with index 0, we go from $S(l_0, l_1, l_2, \emptyset)$ to $S(l_0, l_0 l_1 l_0, l_0 l_2 l_0, (l_0, l_0, I) \cdot \emptyset)$.



Lemma If we flip all edges with index 0, we go from $S(l_0, l_1, l_2, \mathcal{O})$ to $S(l_0, l_0 l_1 l_0, l_0 l_2 l_0, (l_0, l_0, I) \cdot \mathcal{O})$.
 \parallel
 $(l_0, I, l_0) \cdot \mathcal{O}$



Lemma If we flip all edges with index 0, we go from $S = S(l_0, l_1, l_2, \emptyset)$ to $S_0' = S(l_0, l_0 l_1, l_0, l_0 l_2 l_0, (l_0, l_0, I) \cdot \emptyset)$.

Thus, $l_0 \circ S = l_0 \circ S_0' = S(\vec{l}, (I, I, l_0) \cdot \emptyset)$

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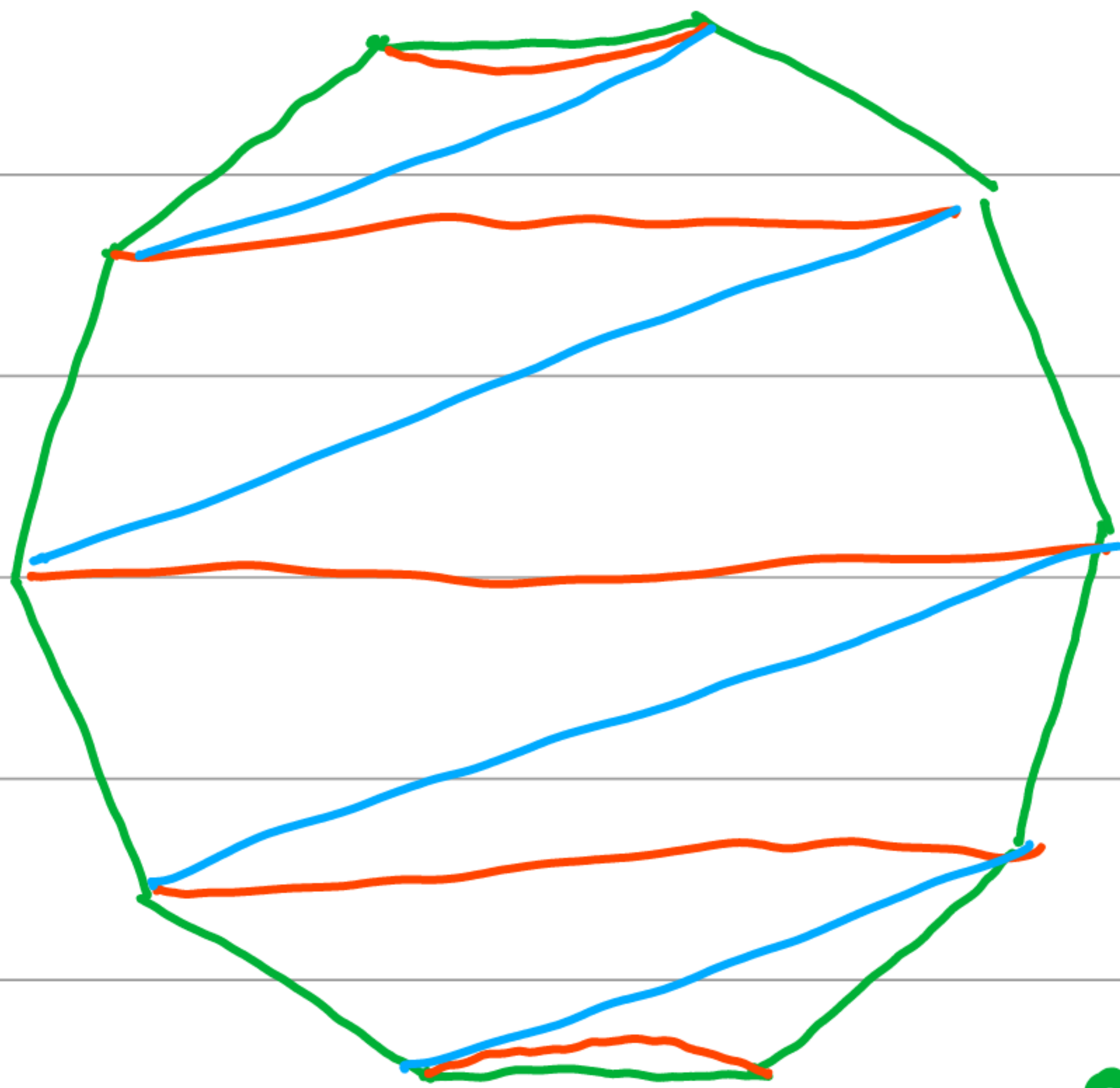
$$l_0 \cdot S = l_0 \cdot S'_0 = S(\vec{l}, (I, I, l_0) \cdot \emptyset)$$
$$l_1 \cdot S = l_1 \cdot S'_1 = S(\vec{l}, (l_1, I, I) \cdot \emptyset)$$
$$l_2 \cdot S = l_2 \cdot S'_2 = S(\vec{l}, (I, l_2, I) \cdot \emptyset)$$

Lemma If we flip all edges with index 0, we go from $S = S(l_0, l_1, l_2, \emptyset)$ to $S'_0 = S(l_0, l_0 l_1, l_0, l_0 l_2 l_0, (l_0, l_0, I) \cdot \emptyset)$.

Thus, $l_0 \cdot S = l_0 \cdot S'_0 = S(\vec{l}, (I, I, l_0) \cdot \emptyset)$
 $l_1 \cdot S = l_1 \cdot S'_1 = S(\vec{l}, (l_1, I, I) \cdot \emptyset)$
 $l_2 \cdot S = l_2 \cdot S'_2 = S(\vec{l}, (I, l_2, I) \cdot \emptyset)$
 $(\mathbb{Z}/2\mathbb{Z})^3$ -action

Ex Deforming Veech's Surfaces.

Regular 10-gon



degenerate

