Symmetric surfaces from dynamics on triangles.

Description

Half-dilation surfaces are fun to build; you can snap together triangles like Magna-tiles®. I will describe a construction of half-dilation surfaces built from triangles produced by a $(\mathbb{Z}/2\mathbb{Z})(\mathbb{Z}/2\mathbb{Z})(\mathbb{Z}/2\mathbb{Z})$ action on homothety-equivalence classes of triangles in the plane. The advantage of this construction is that it produces surfaces with non-elementary Veech groups. Some of the surfaces that arise have infinite type, some others are already well-known: the Bouw-Möller lattice surfaces. This talk is about joint work with Seth Foster and Zhi Heng Liu.

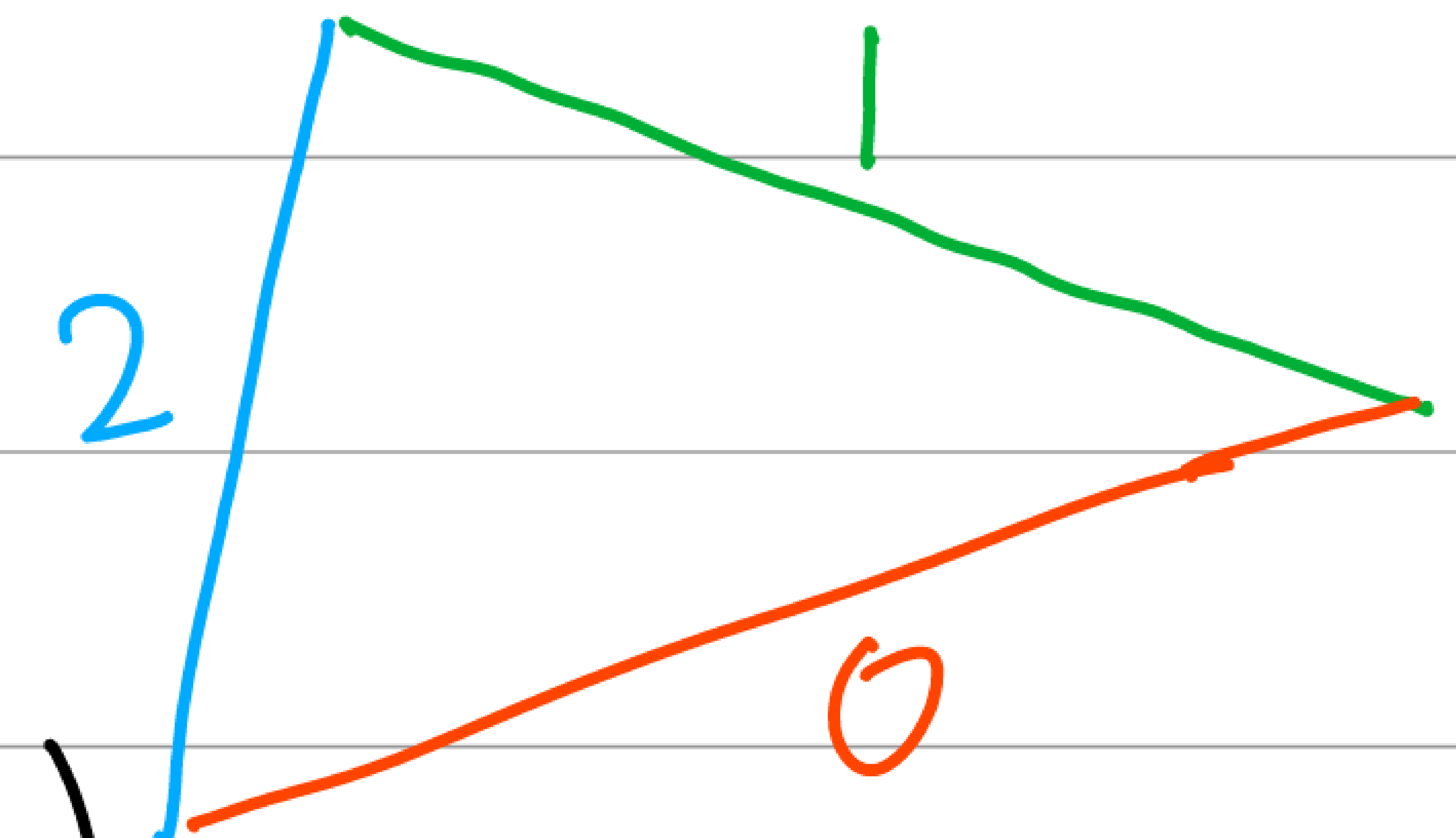
Translation, Dilation, Affine and other Structures on Surfaces

Apr 7 – 11, 2025
Institut de Mathématiques de Toulouse

Enter your sea

I. Triangles

A marked triangle is a triangle in $\mathbb{R}^2$ with edges labeled by $\{0, 1, 2\}$.



Associated to a marked triangle is its triple of slopes $\vec{m} = (m_0, m_1, m_2)$ in $\hat{\mathbb{R}}^3$ where $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

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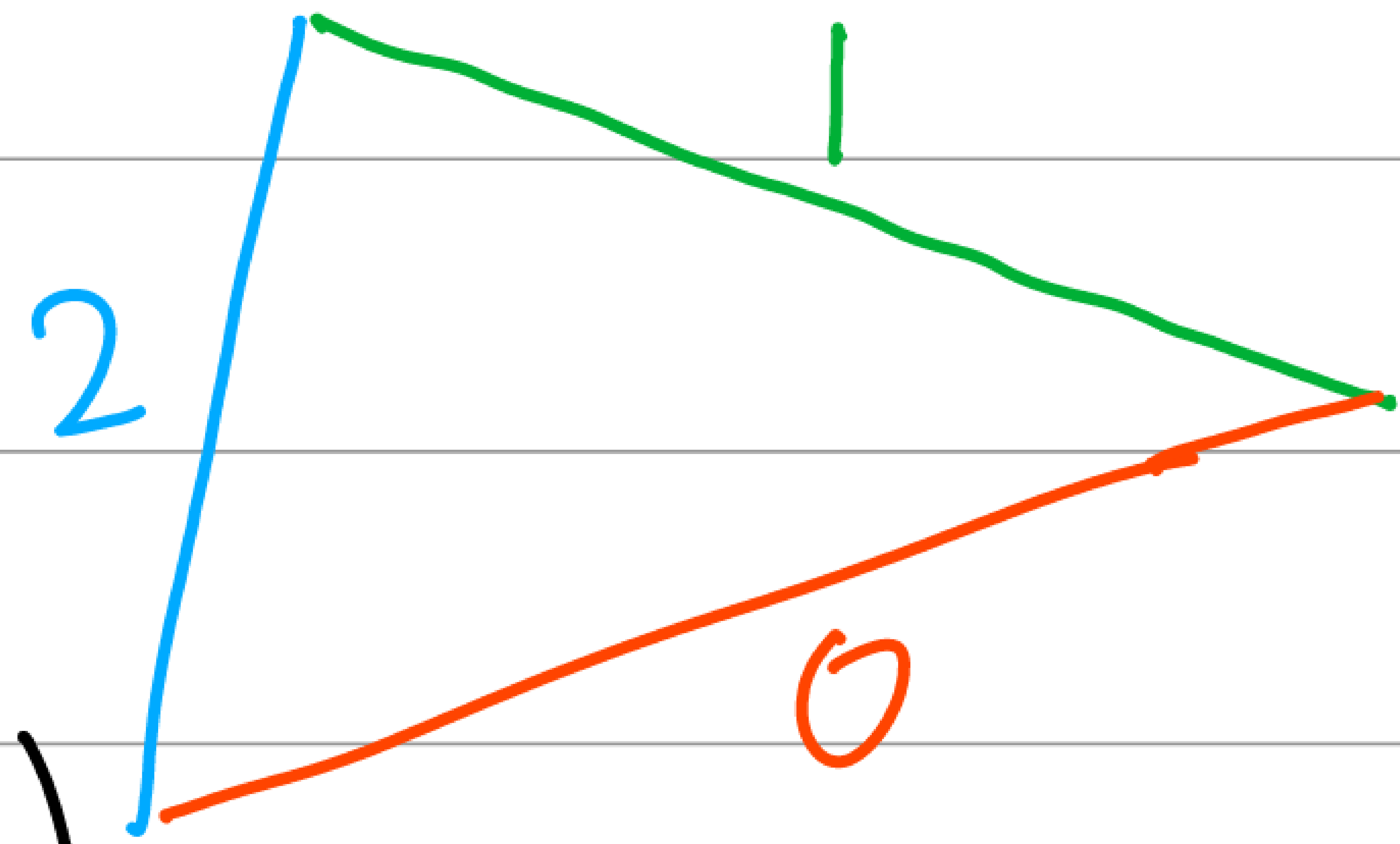

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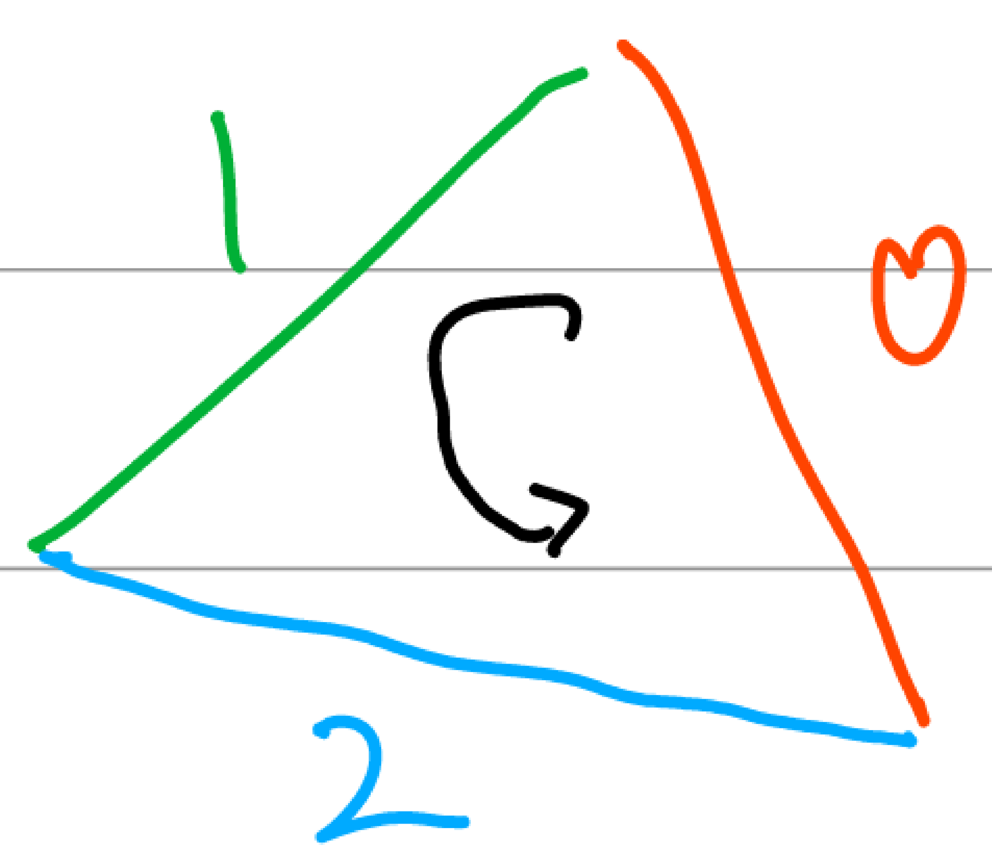
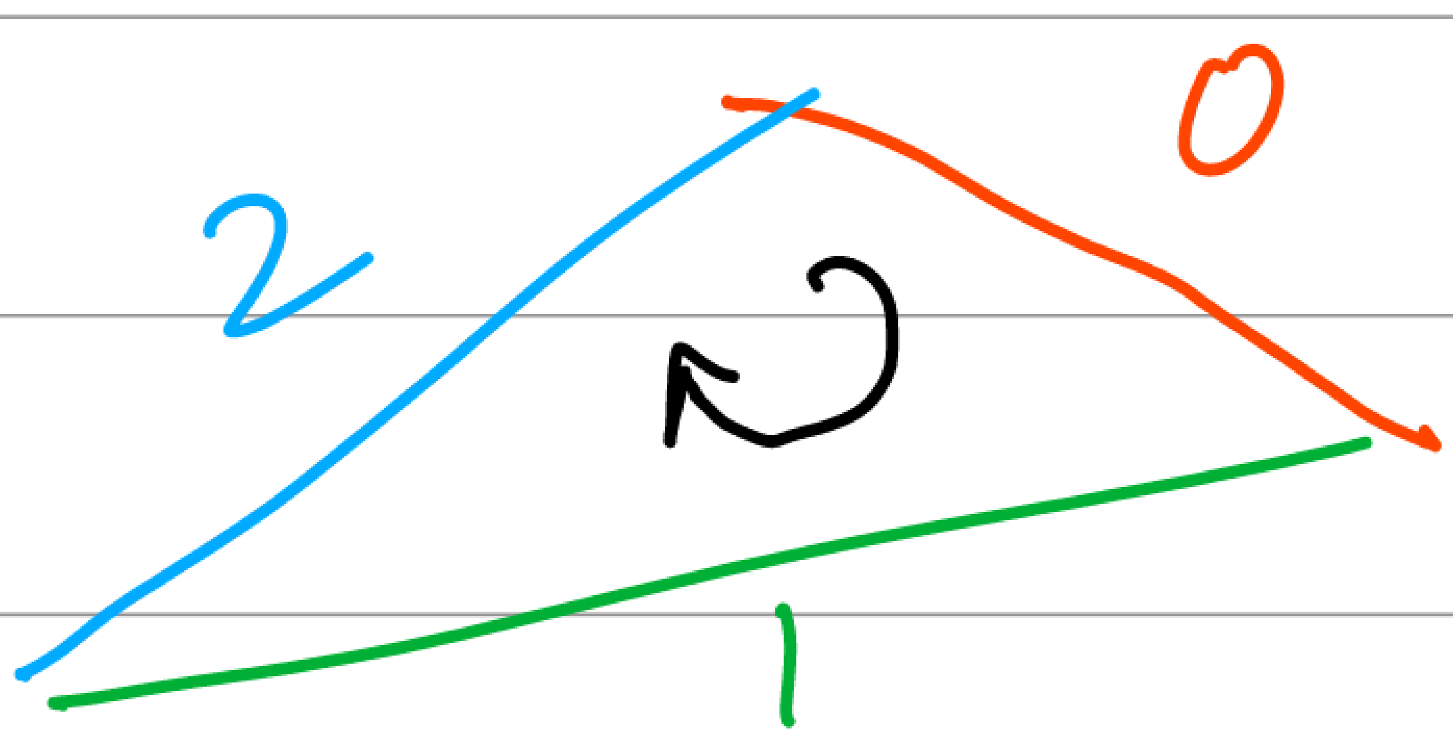
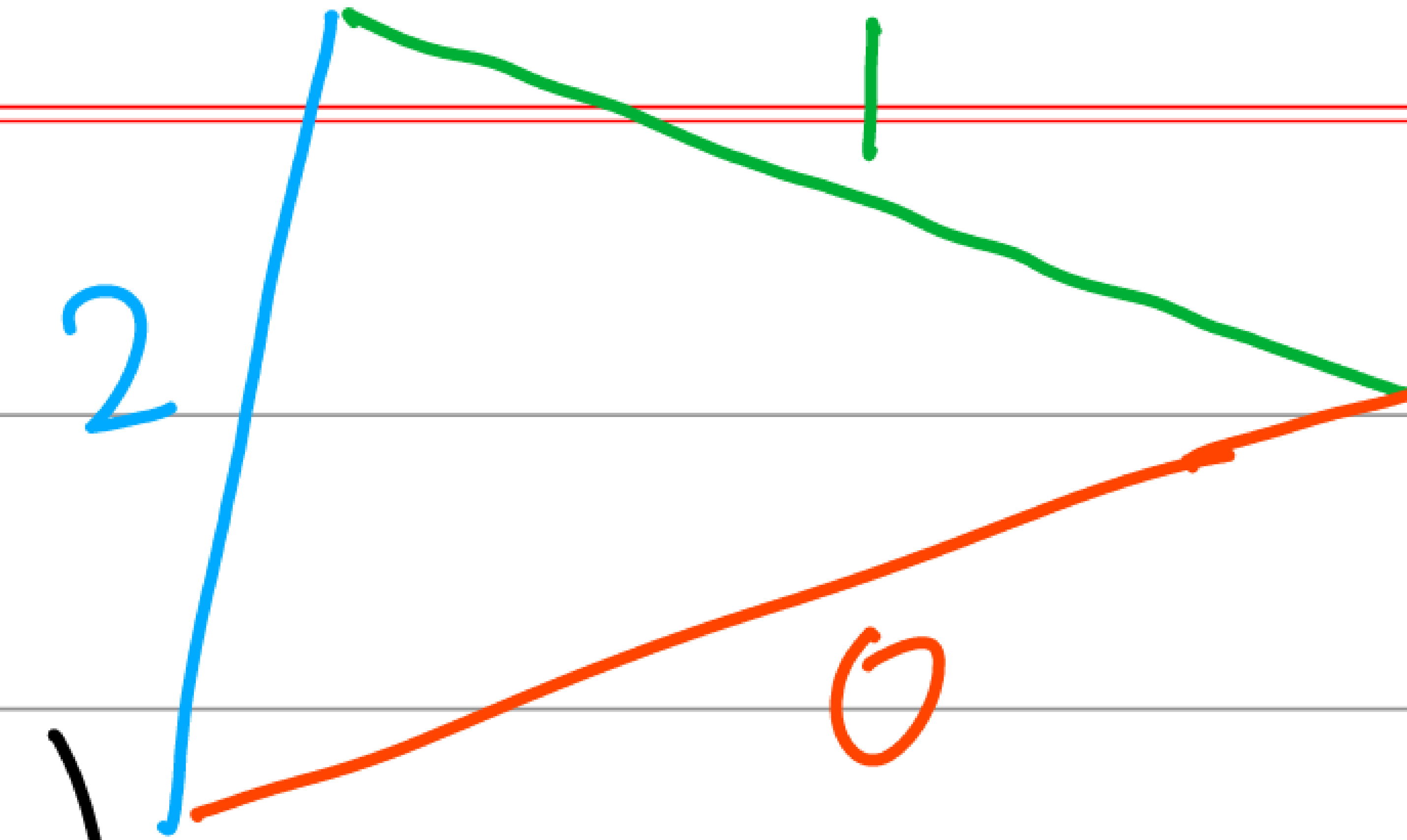




positively oriented or negatively oriented

Depends on cyclic order of edge labels.

Associated to a marked triangle is its triple of slopes $\vec{m} = (m_0, m_1, m_2)$ in $\widehat{\mathbb{R}}^3$ where $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.



The homothety or half-dilation group is

$$HD = \{ z \mapsto az + b : a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{C} \} \subset \mathrm{Aff}(\mathbb{C}).$$

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Obs. The map sending a marked triangle in the plane to its triple $\vec{m} \in \widehat{\mathbb{R}}^3$ induces a bijection from triangles up to HD to triples of distinct elements of $\widehat{\mathbb{R}}$.

Furthermore, a triangle is positively oriented iff its slope triple is in decreasing cyclic order.

II. Quadrilaterals.

Setup: $GL(2,\mathbb{R}) / \underbrace{H \cap GL(2,\mathbb{R})}_{(\mathbb{R}\setminus\{0\},\times)} = PGL(2,\mathbb{R})$

$= \text{Möb}(\mathbb{R})$

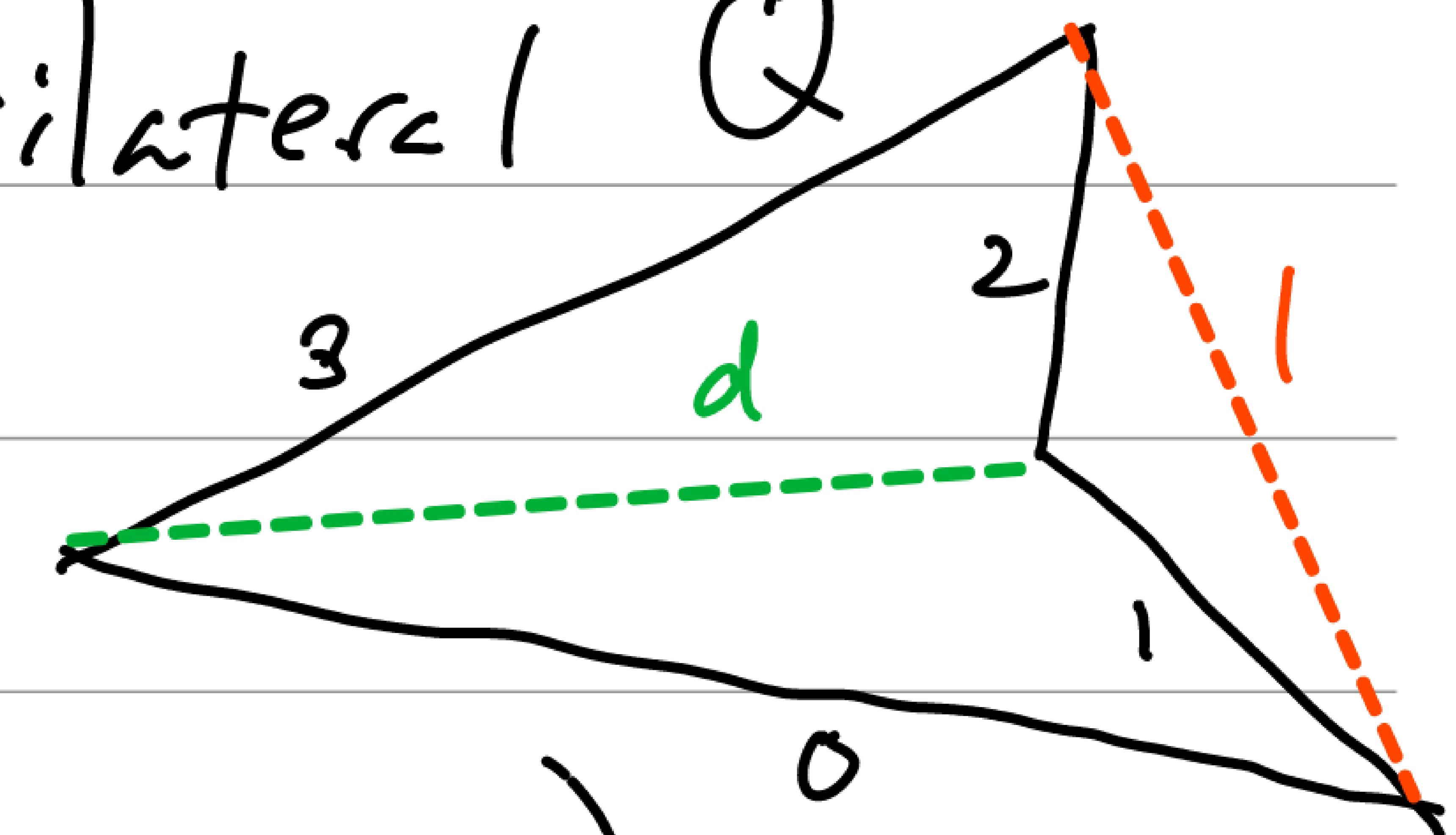
$= \text{Isom}(\mathbb{H})$

where $\text{Möb}(\mathbb{R}) = \left\{ z \mapsto \frac{az+b}{cz+d} : \begin{array}{l} a,b,c,d \in \mathbb{R}, \\ ad-bc \neq 0 \end{array} \right\}$

is the isometry group of the hyperbolic plane $\mathbb{H} = (\mathbb{C} \setminus \mathbb{R}) / \text{complex conjugation}$.

(We identify the upper and lower half planes via $z \mapsto \bar{z}$.)

Consider a marked quadrilateral Q with diagonals D and d.



Quad. Thm 1 If $(m_0, m_1, m_2, m_3; m_D, m_d)$ is the slope data of any quadrilateral Q that is non-degenerate in the sense that $m_k \neq m_{k+1 \pmod 4}$ $\forall k \in \{0,1,2,3\}$,

then $\exists$ an involution $\iota_Q = \dfrac{az+b}{cz-a} \in \text{Möb}(\mathbb{R})$

s.t. $m_0 \overset{\iota_Q}{\longleftrightarrow} m_2$, $m_1 \overset{\iota_Q}{\longleftrightarrow} m_3$, $m_D \overset{\iota_Q}{\longleftrightarrow} m_d$.

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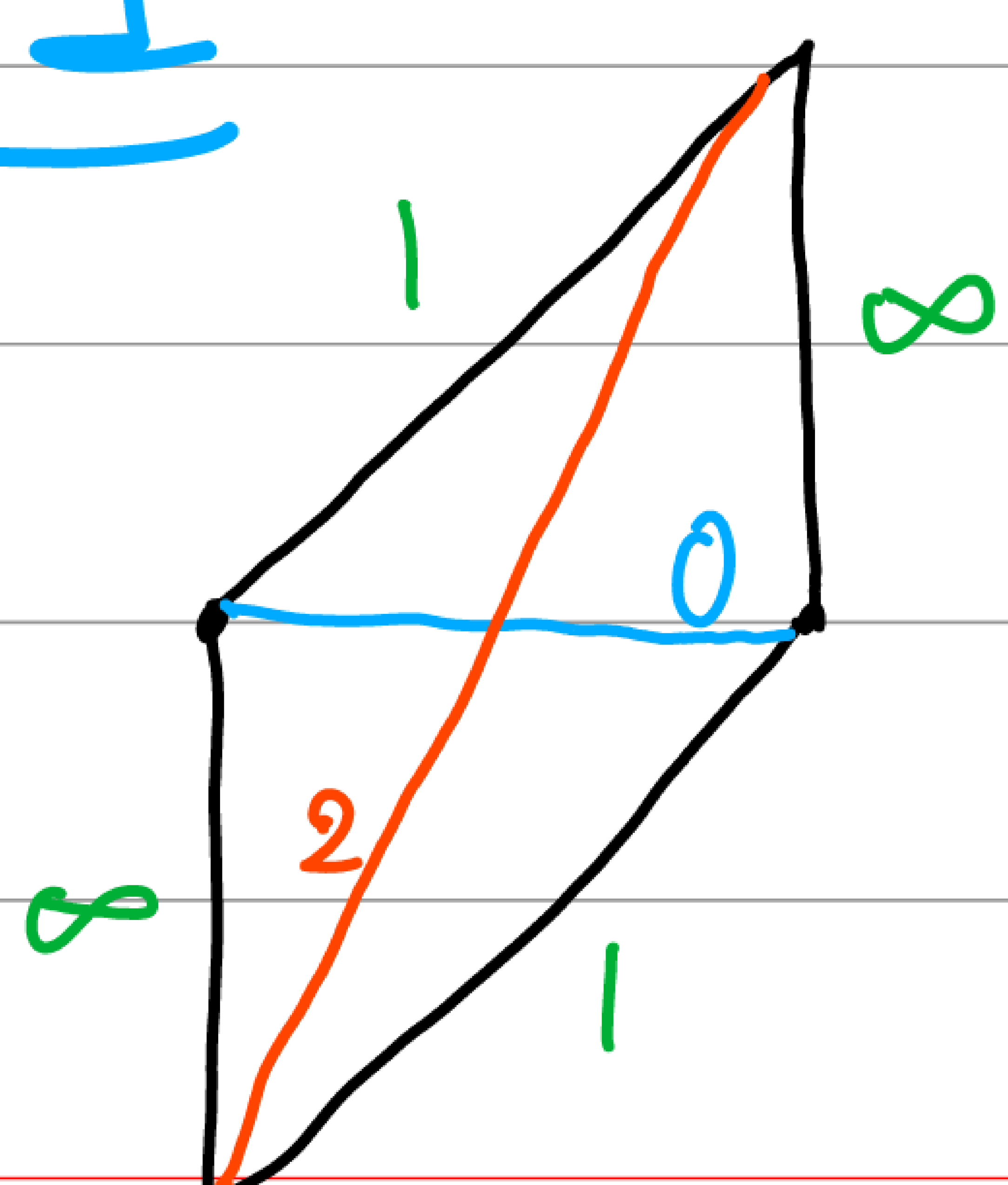
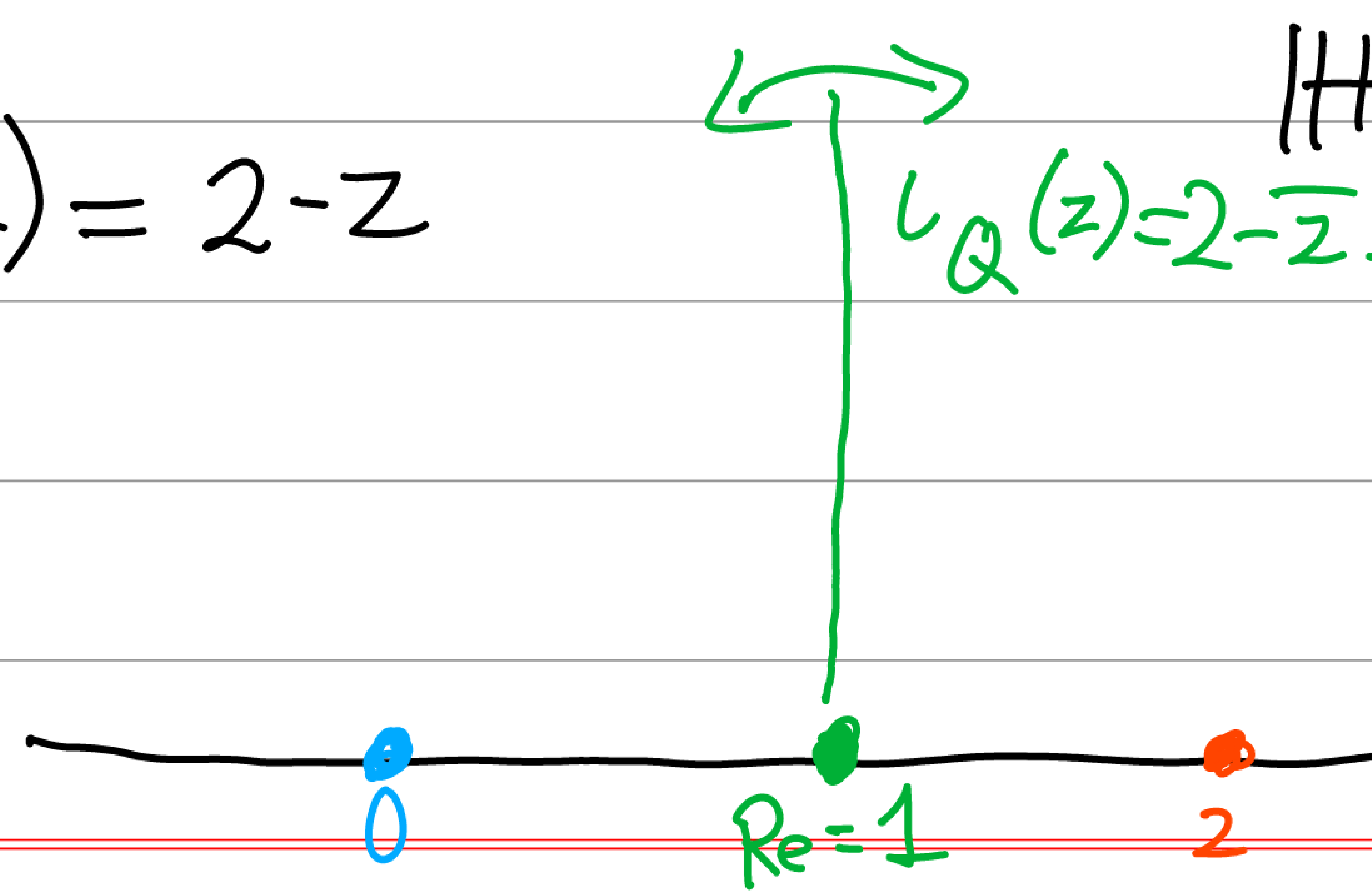
Ex 1:



$\iota_Q(z) = 2 - z$



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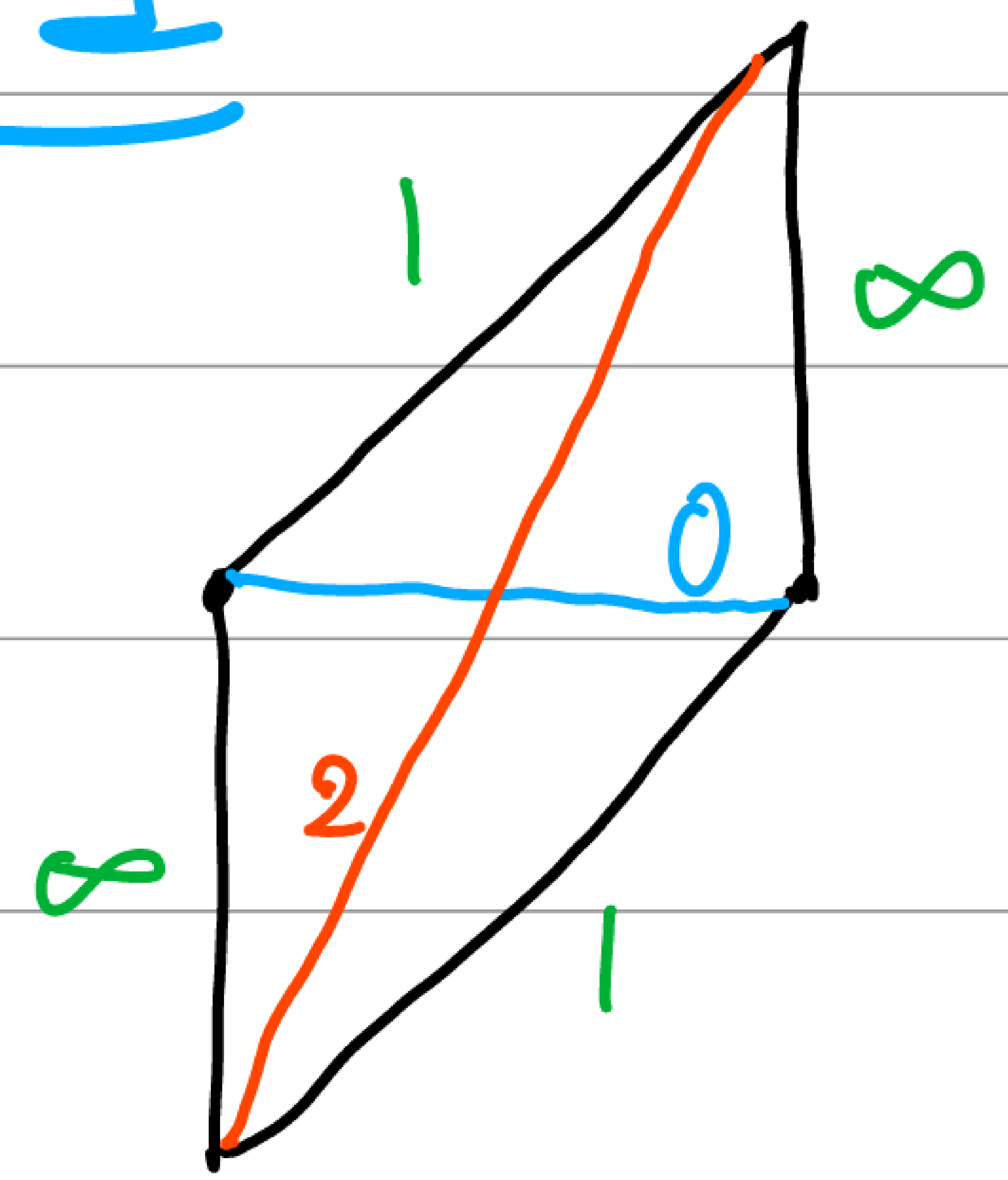

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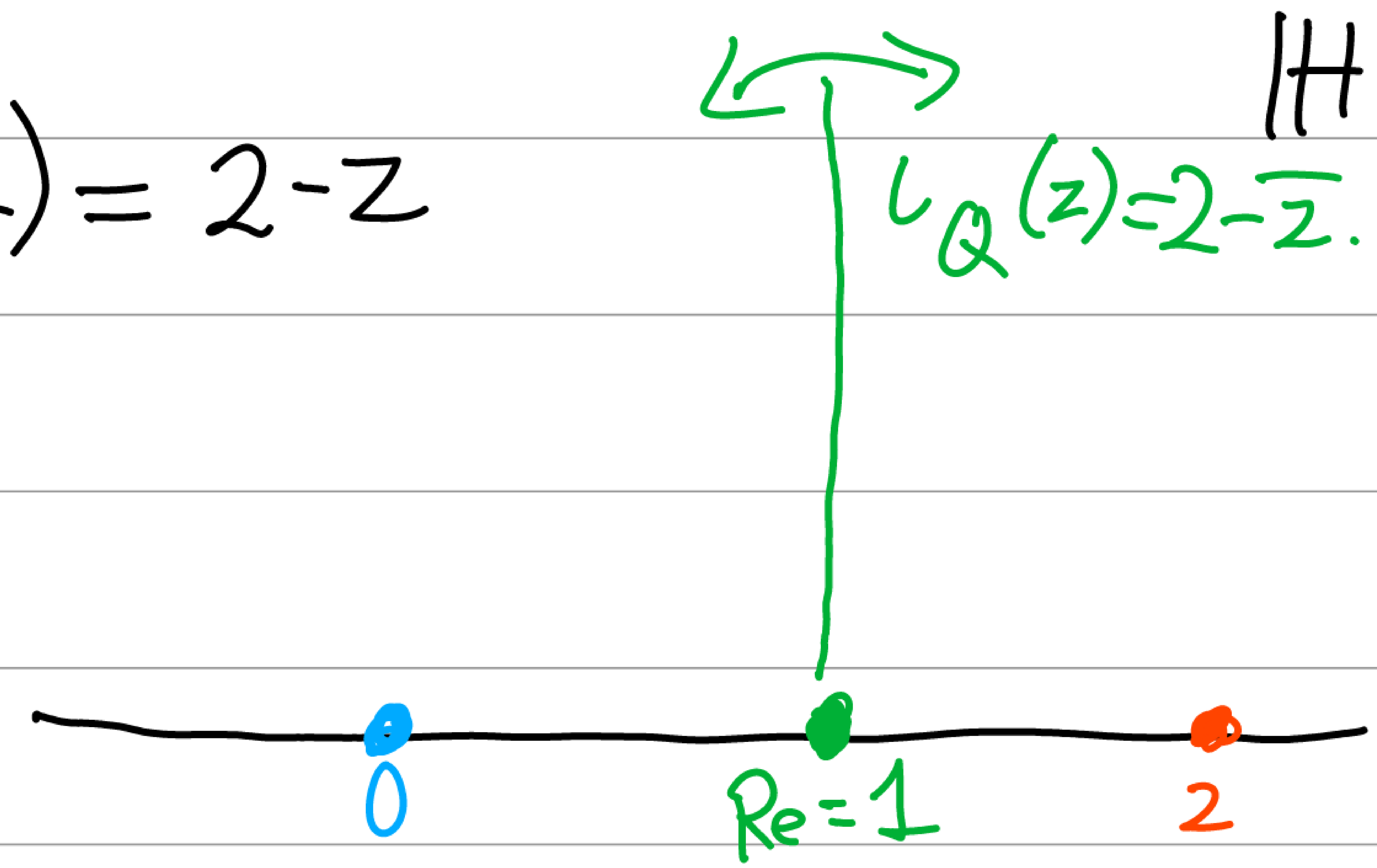
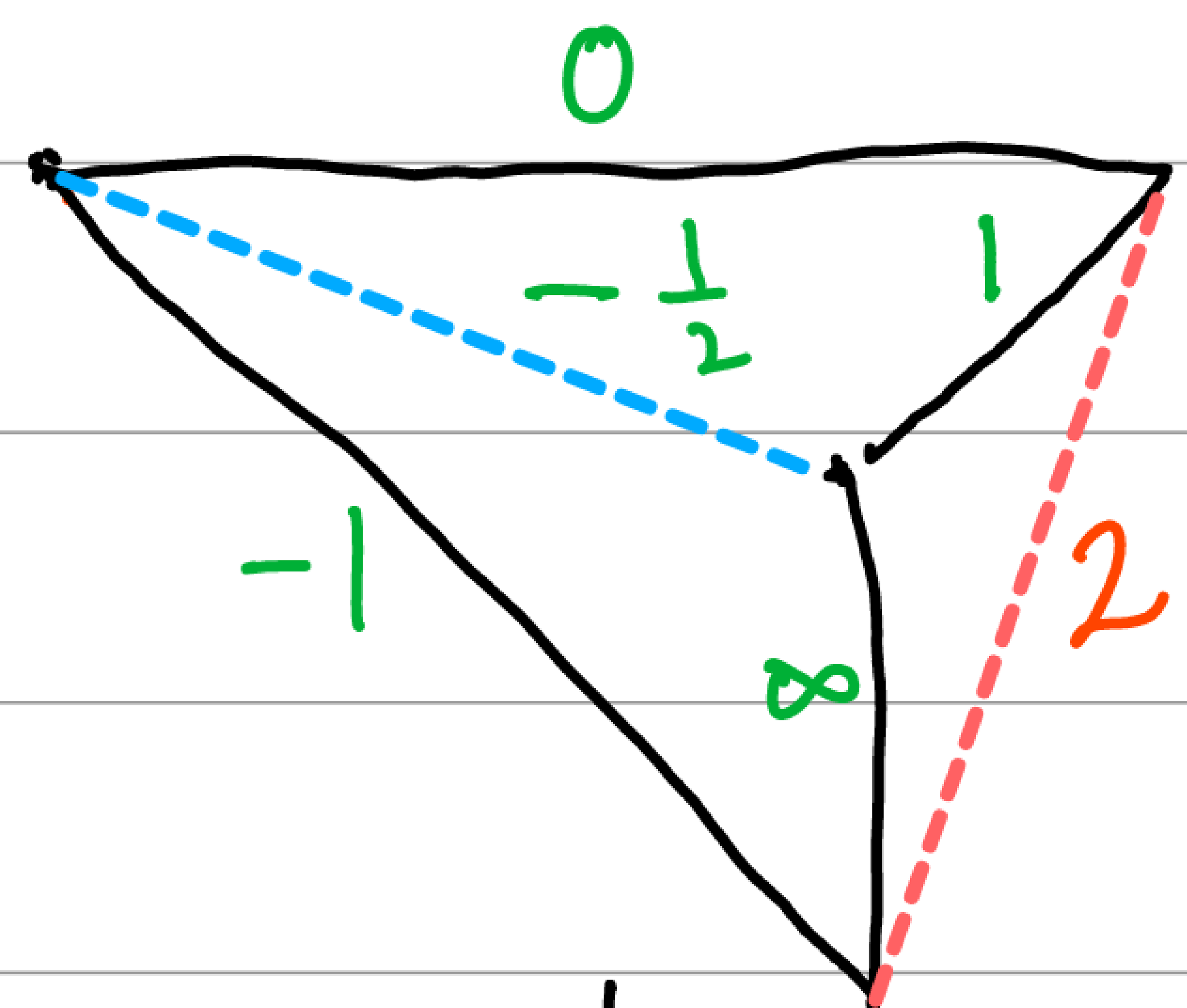
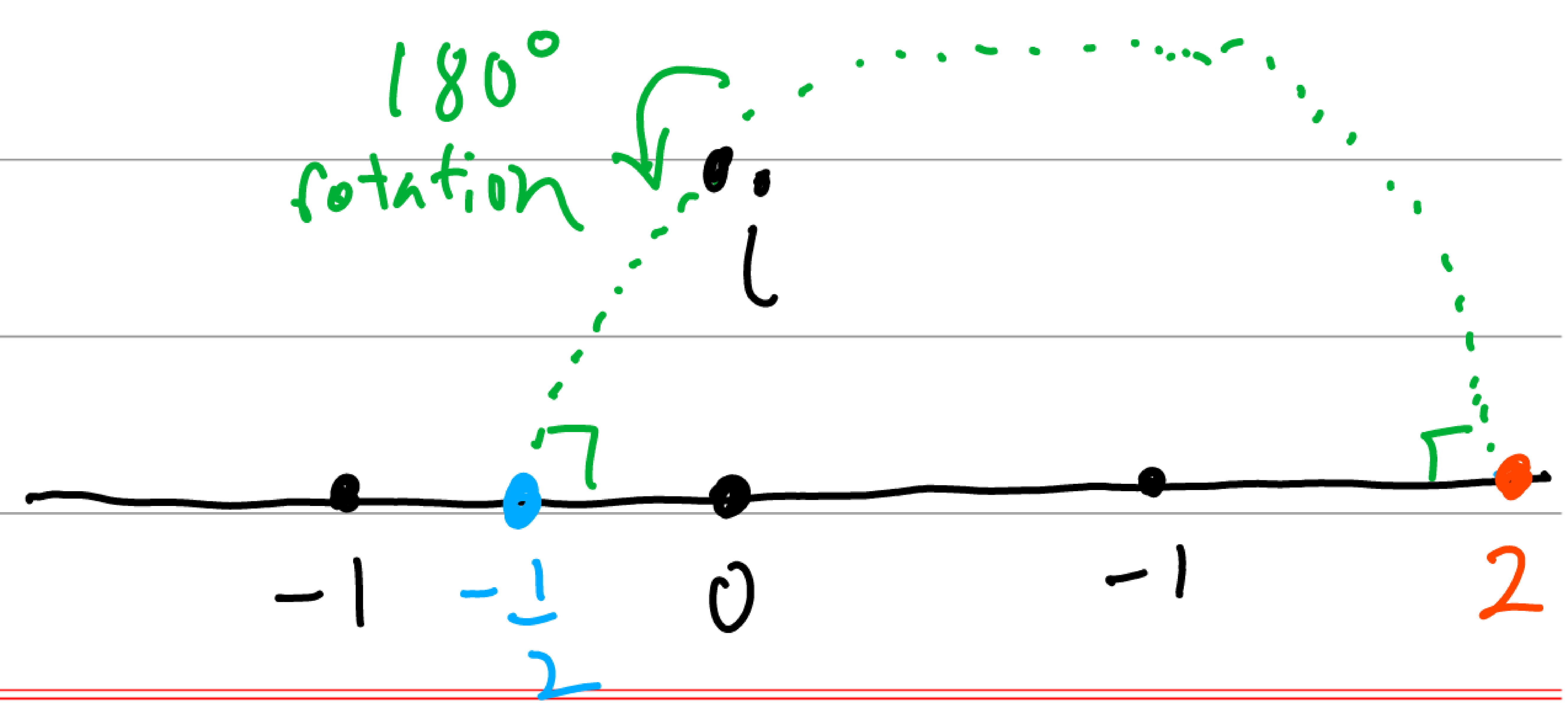


Ex 2

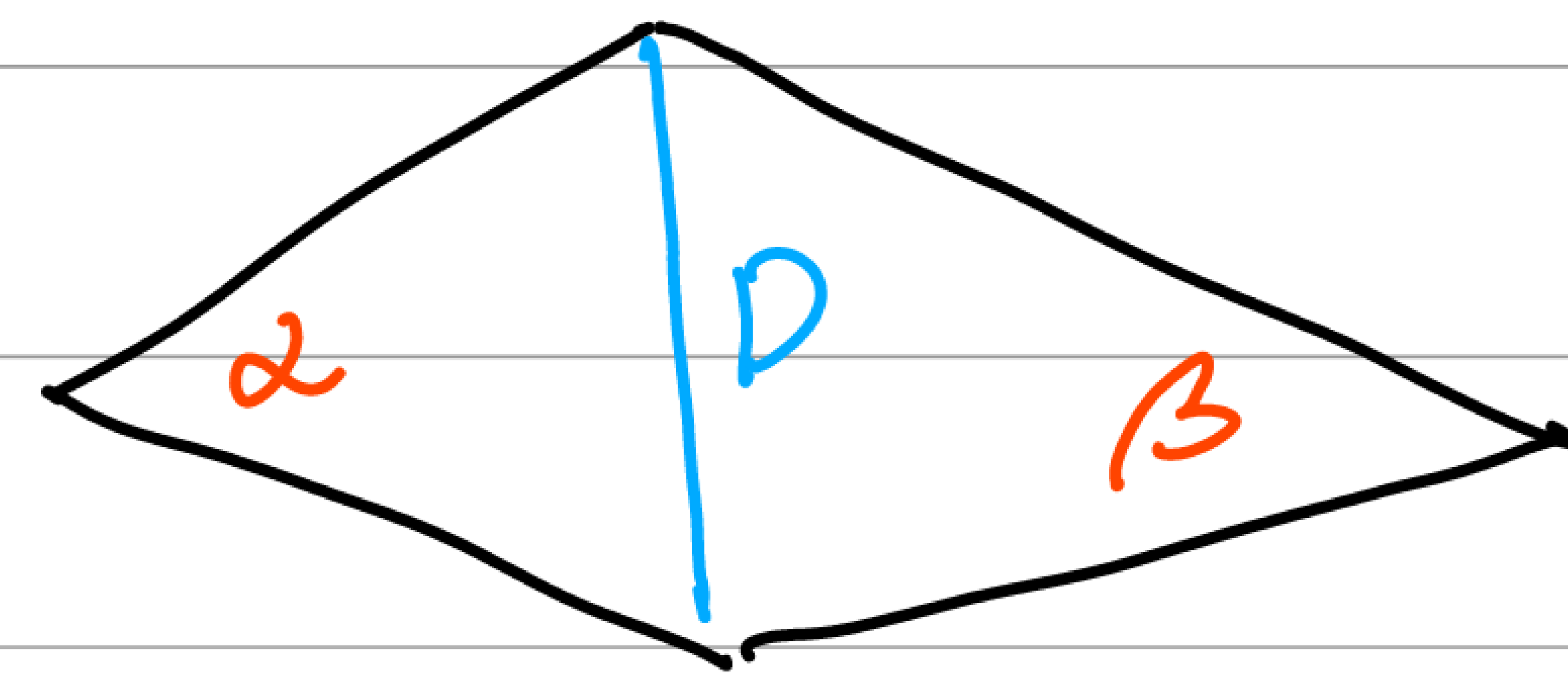


$$\iota_Q(z) = \frac{-1}{z}$$



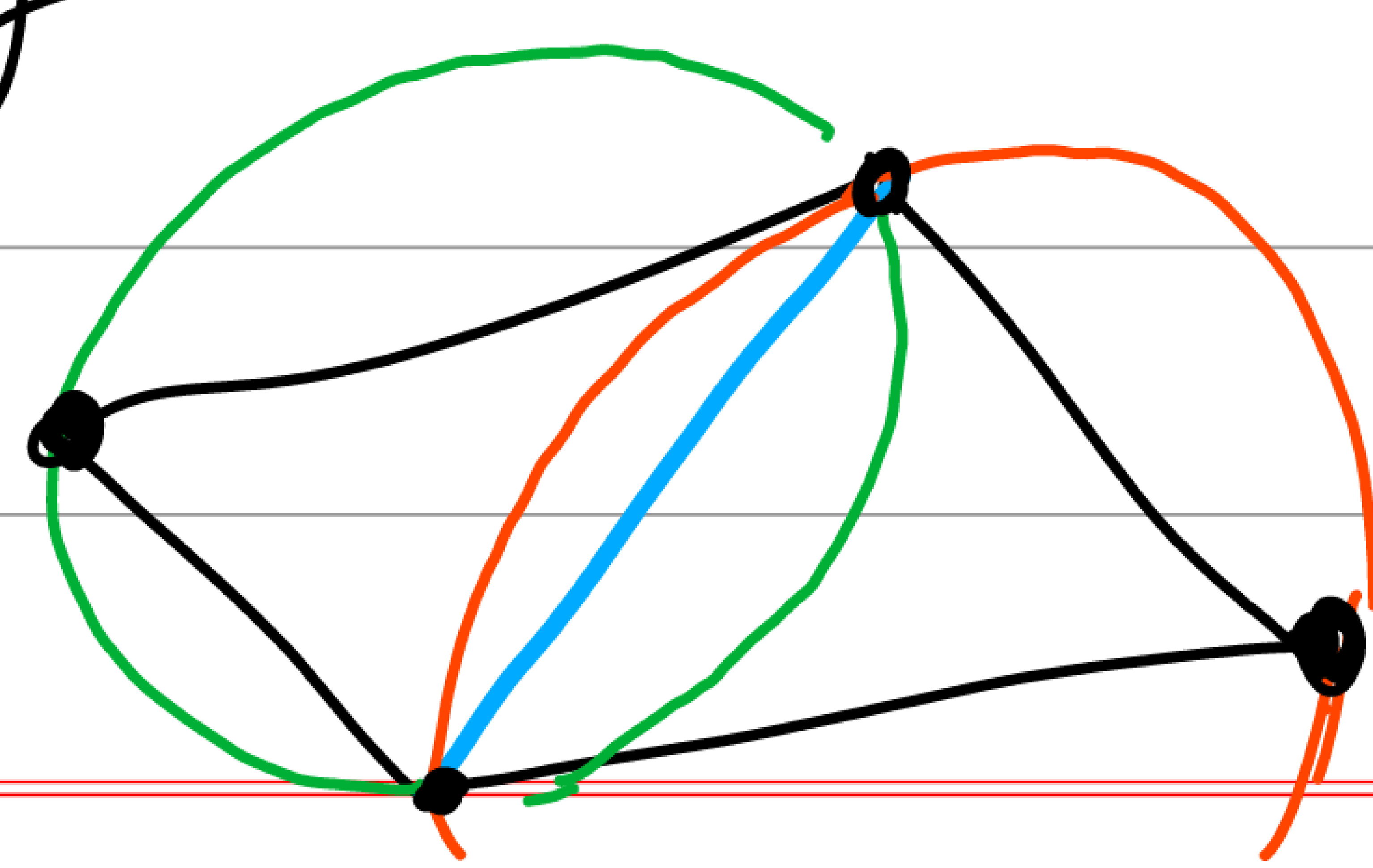
Quad Thm 2 A non-degenerate quadrilateral Q is convex iff ι_Q is orientation reversing on both $\hat{\mathbb{R}}$ and $\mathbb{H}$.

Now assume Q is convex. A diagonal D is Delaunay if opposite angles sum to $\leq \pi$.



$\alpha + \beta \leq \pi$

(Equivalently, circles inscribing the resulting triangles do not contain the remaining vertex of Q in their interiors.)



Quad Thm 3: Let Q be strictly convex.

Let $\gamma = \mathrm{Fix}(\iota_Q) \subset \overline{\mathbb{H}}$ which is a geodesic.

Then D is Delaunay iff either

$i \in \gamma$ or i lies on the same side of γ as m_D.

case when Q is inscribed in a circle.

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Let $\tau \in \mathrm{M\ddot{o}b}(\mathbb{R}) = PSL(2,\mathbb{R})$. Then

$\tau(D)$ is diagonal in $\tau(Q)$ iff

$\tau^{-1}(i) \in \gamma$ or if $\tau^{-1}(i)$ lies on the

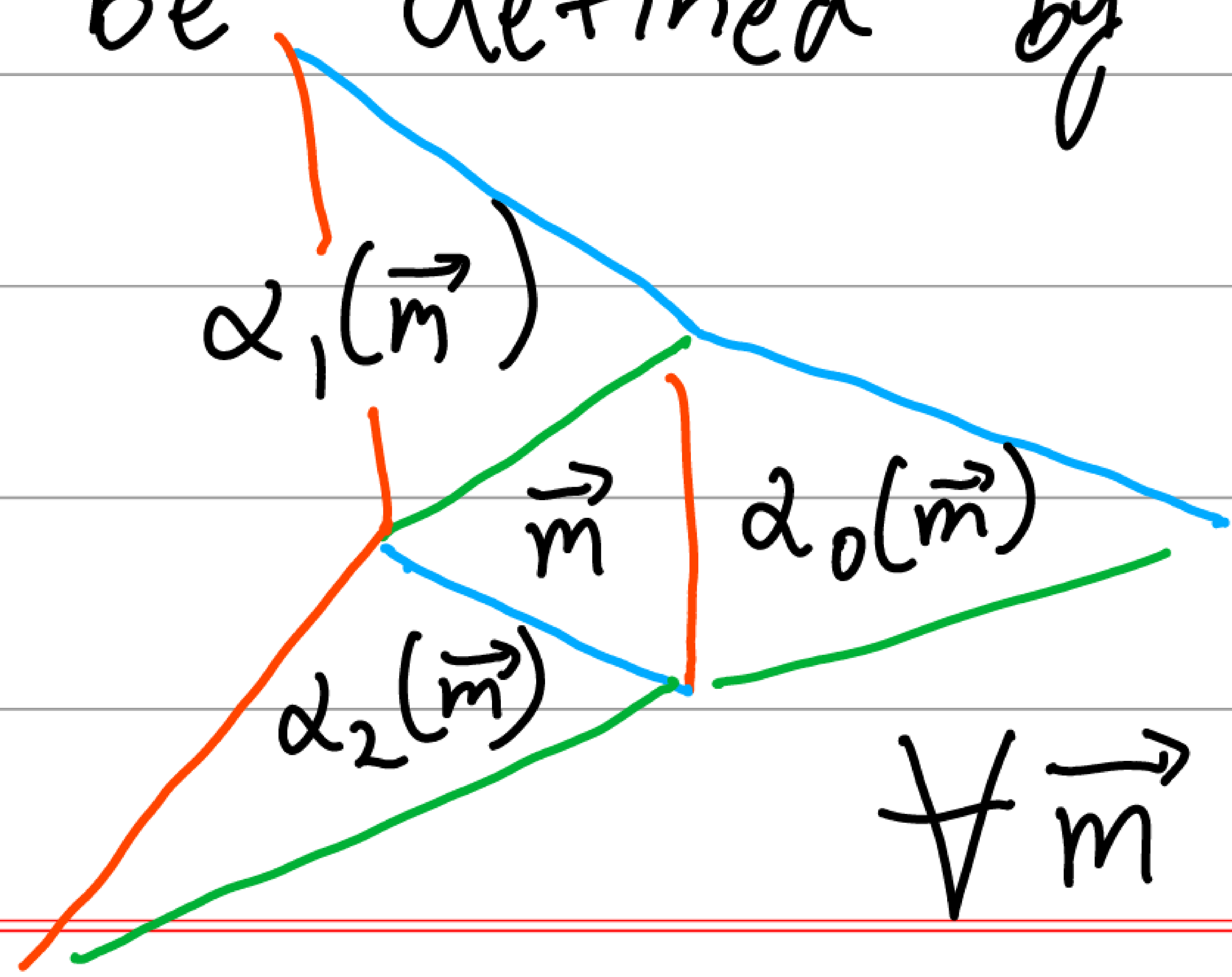
same side of γ as m_D.

III. Constructing Surfaces

Choose a triple of orientation-reversing Möbius involutions $\vec{\iota} = (\iota_0, \iota_1, \iota_2)$.

For $i \in \{0,1,2\}$, let $\alpha_i : \hat{\mathbb{R}}^3 \to \hat{\mathbb{R}}^3$

be defined by $\alpha_i(\vec{m})_j = \begin{cases} \iota_i(m_j) & \text{if } i \neq j \\ m_j & \text{if } i = j. \end{cases}$



$G = \langle \alpha_0, \alpha_1, \alpha_2 \rangle$,

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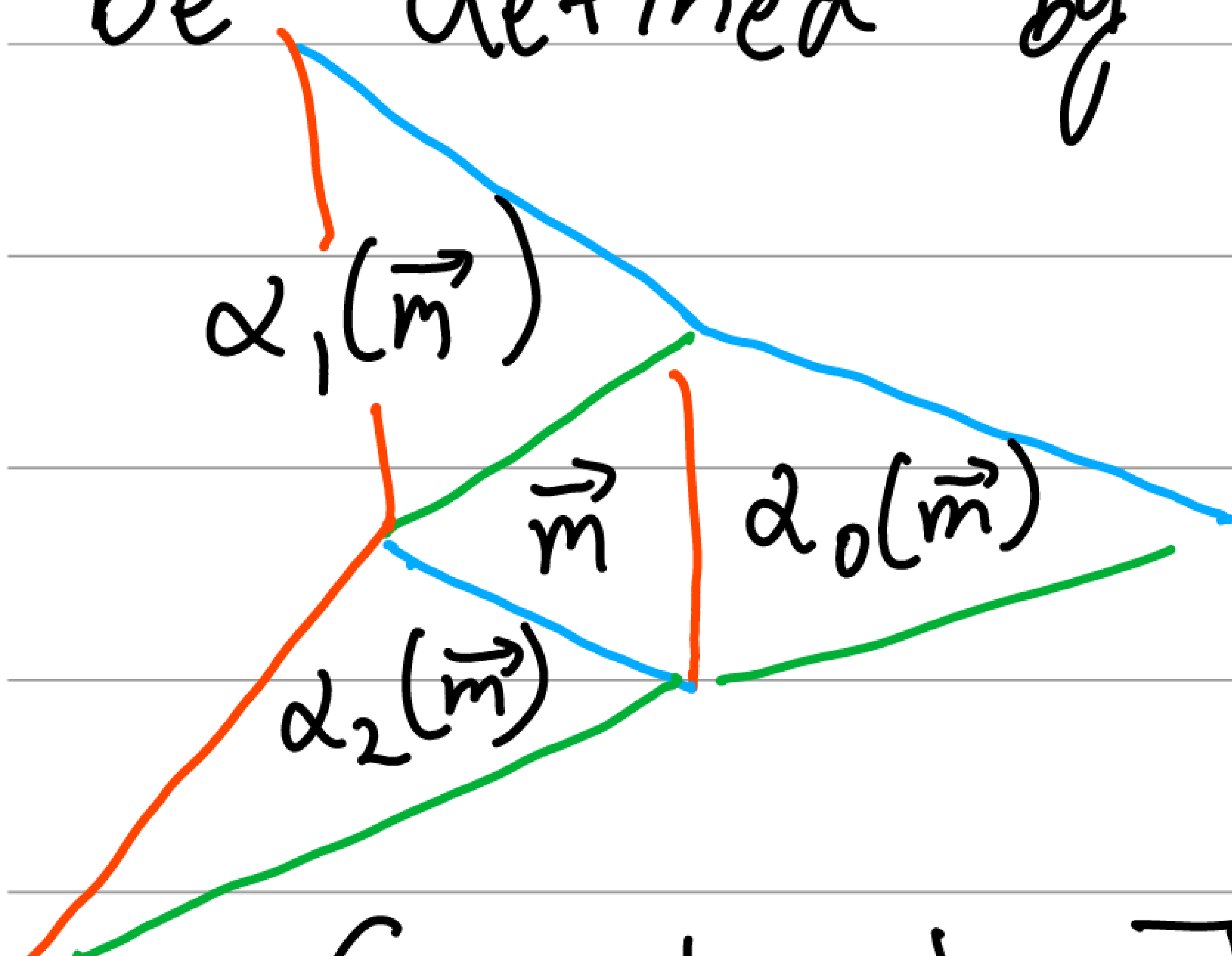

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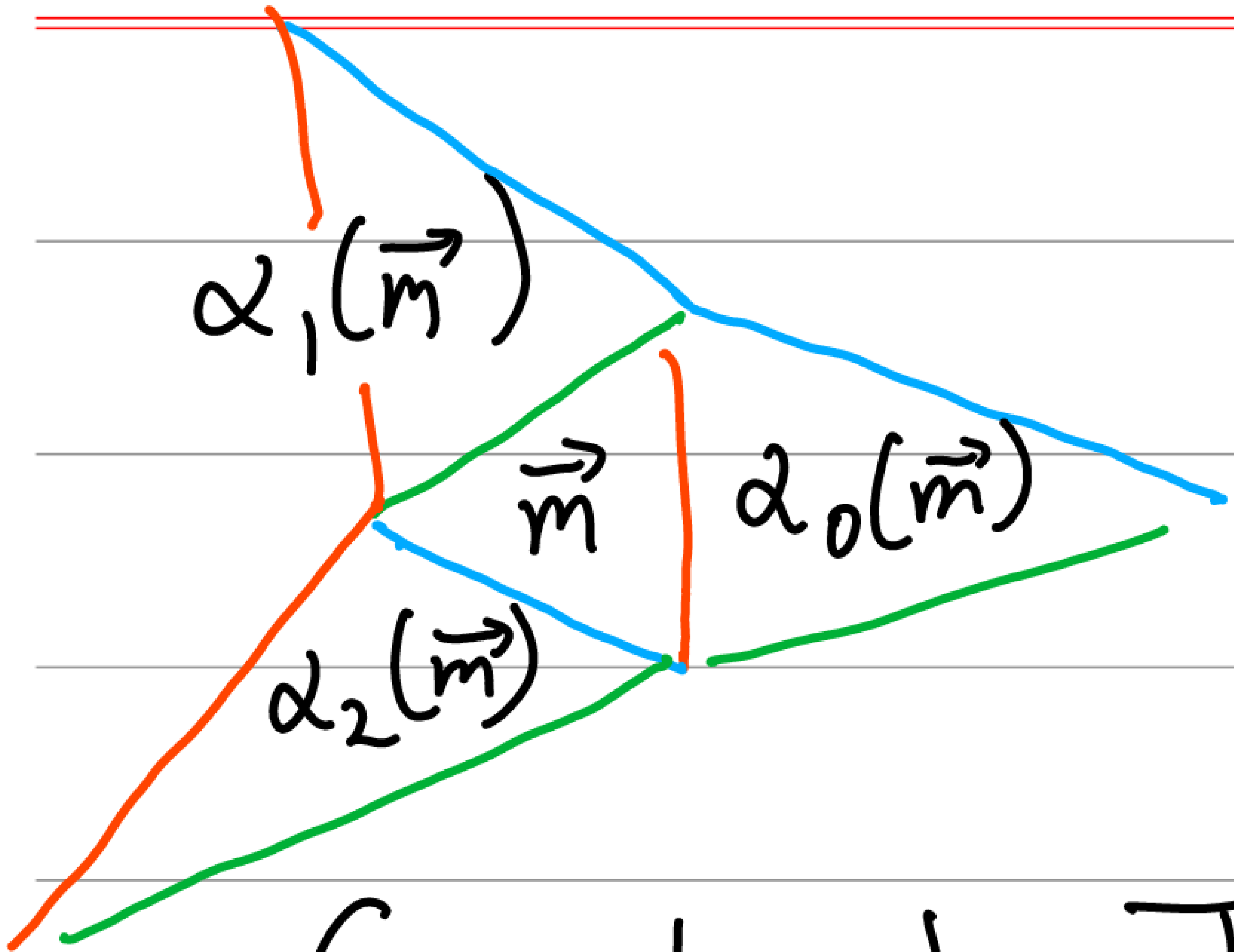
Glue edge i of $T_{\vec{m}}$ to edge i of $T_{\alpha_i(\vec{m})}$.



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Def We call $S = S(\vec{\iota}, \mathcal{O})$ <u>successful</u> if all triangles produced have the same orientation. (We allow some degeneracy.)

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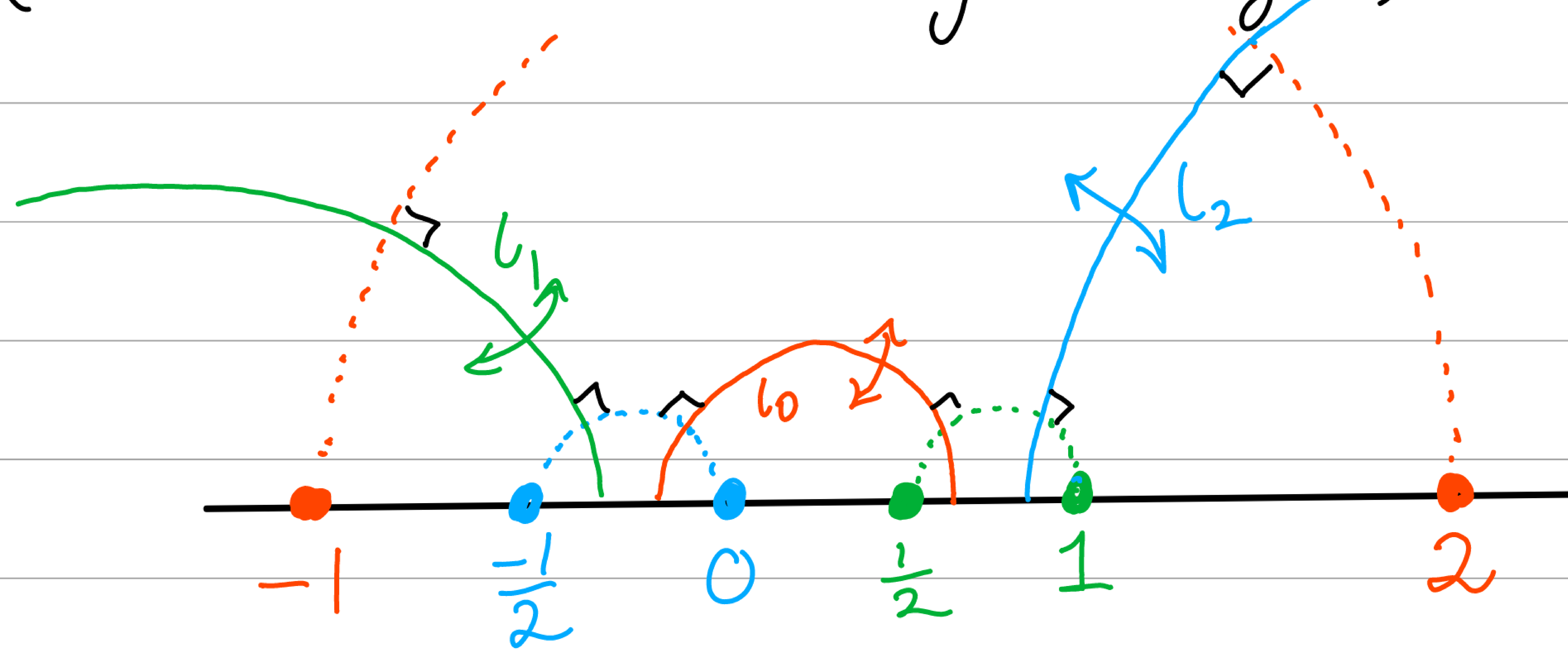
Ex

Red: 0
Green: 1
Blue: 2



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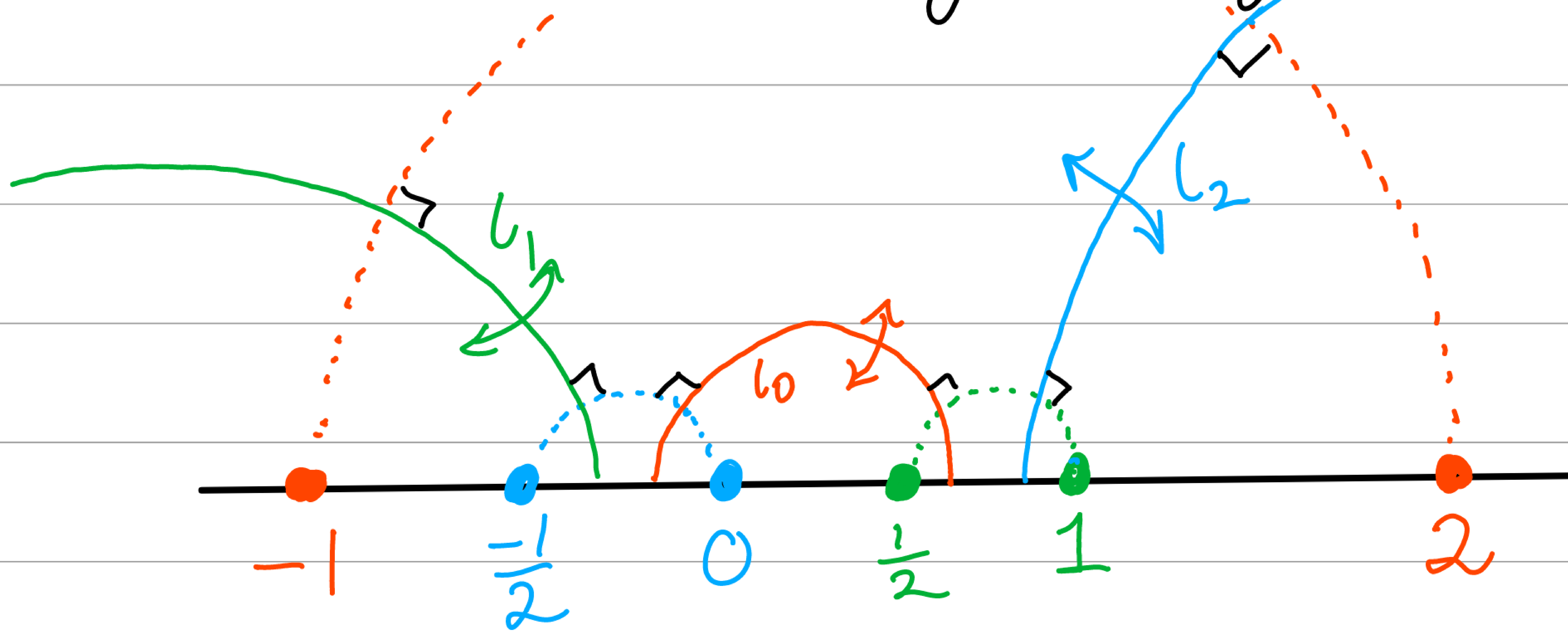

This construction will be successful because all triples (red, green, blue) decrease.

Ex

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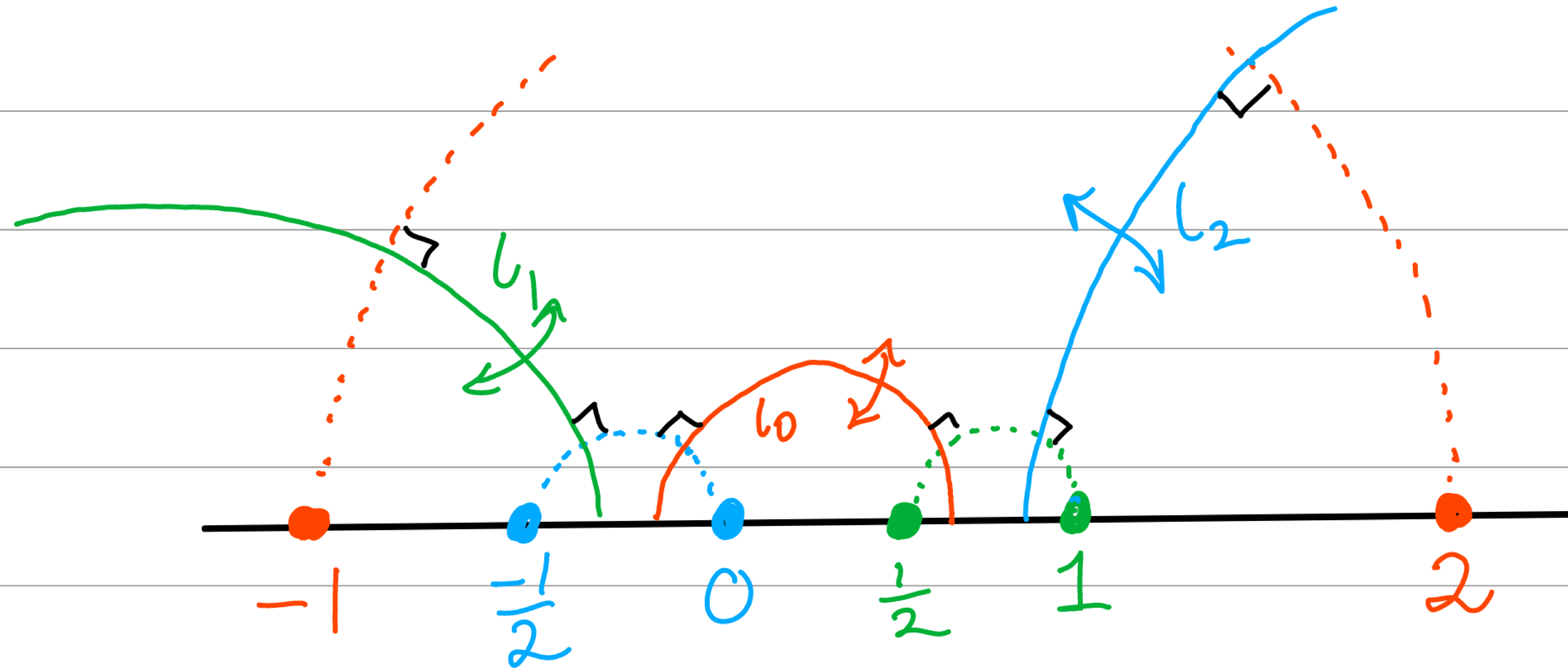
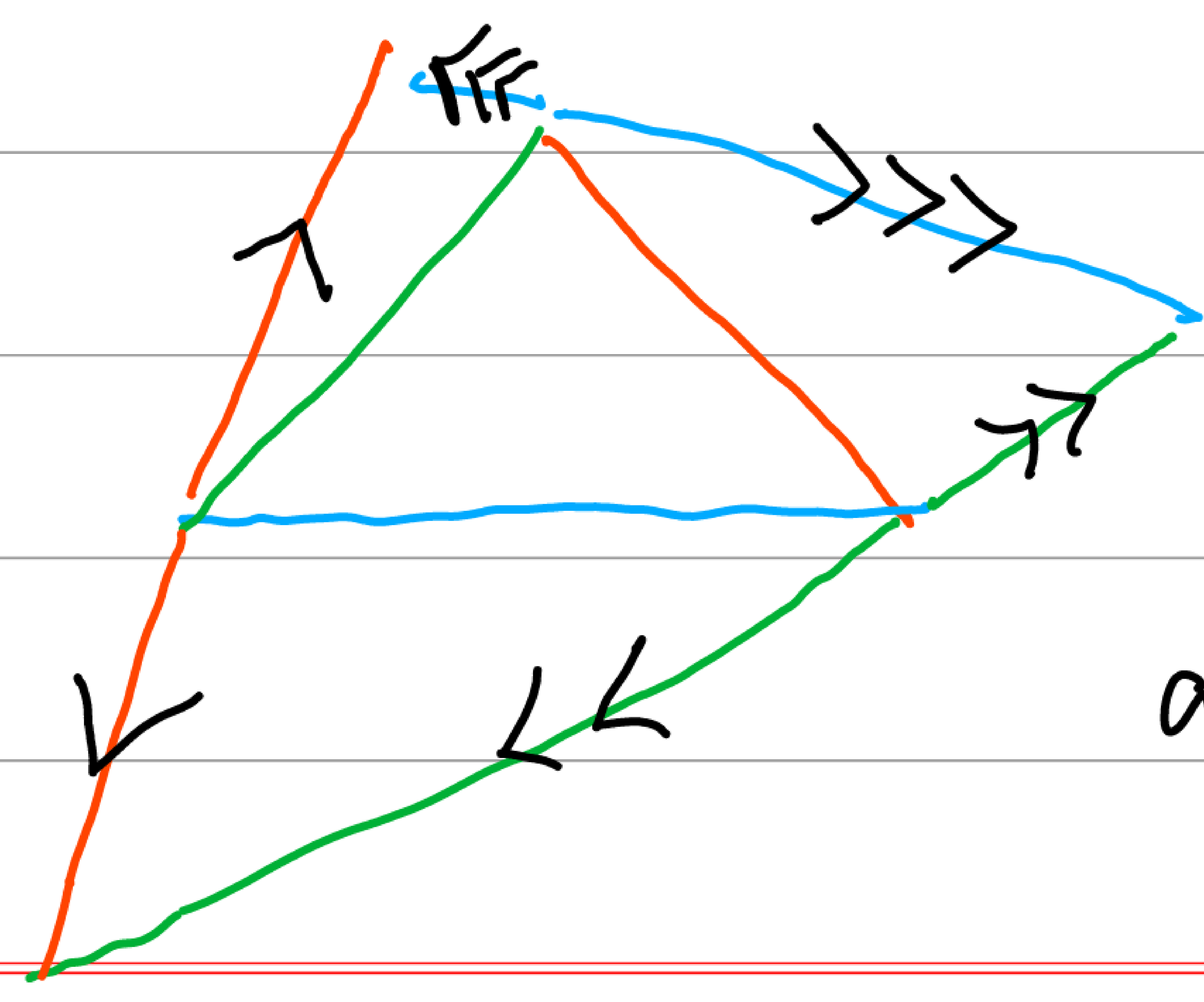
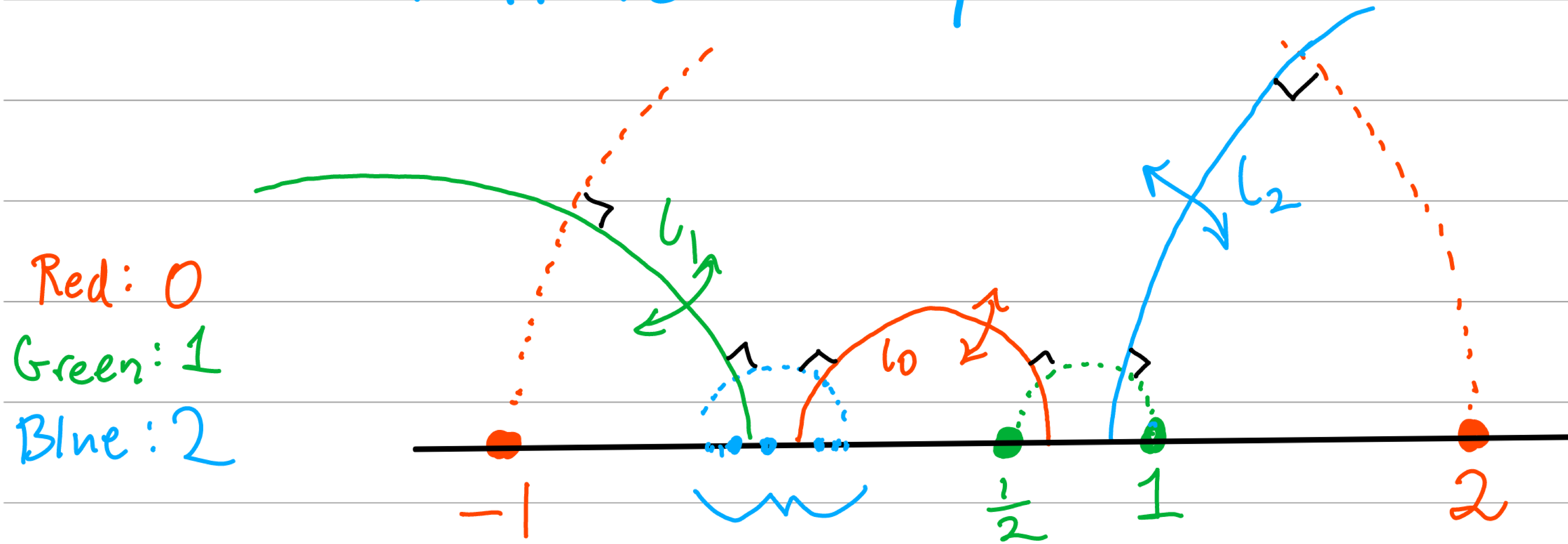


$\mathcal{O} = G(-1, 0, 1)$. $S = S(\vec{\iota}, \mathcal{O})$:



Studied by Taro Shima arXiv:2205.00100

Related infinite example:



Thm The Veech group of a successfully constructed $S = S(\vec{\iota}, \theta)$ contains a subgroup of $\langle \iota_0, \iota_1, \iota_2 \rangle$ of at most index 8.

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Thm If $\mathcal{O}$ is finite, the orbit $PSL(2,\mathbb{R}) \cdot S$ is closed in the space of triangulable half-dilation surfaces.

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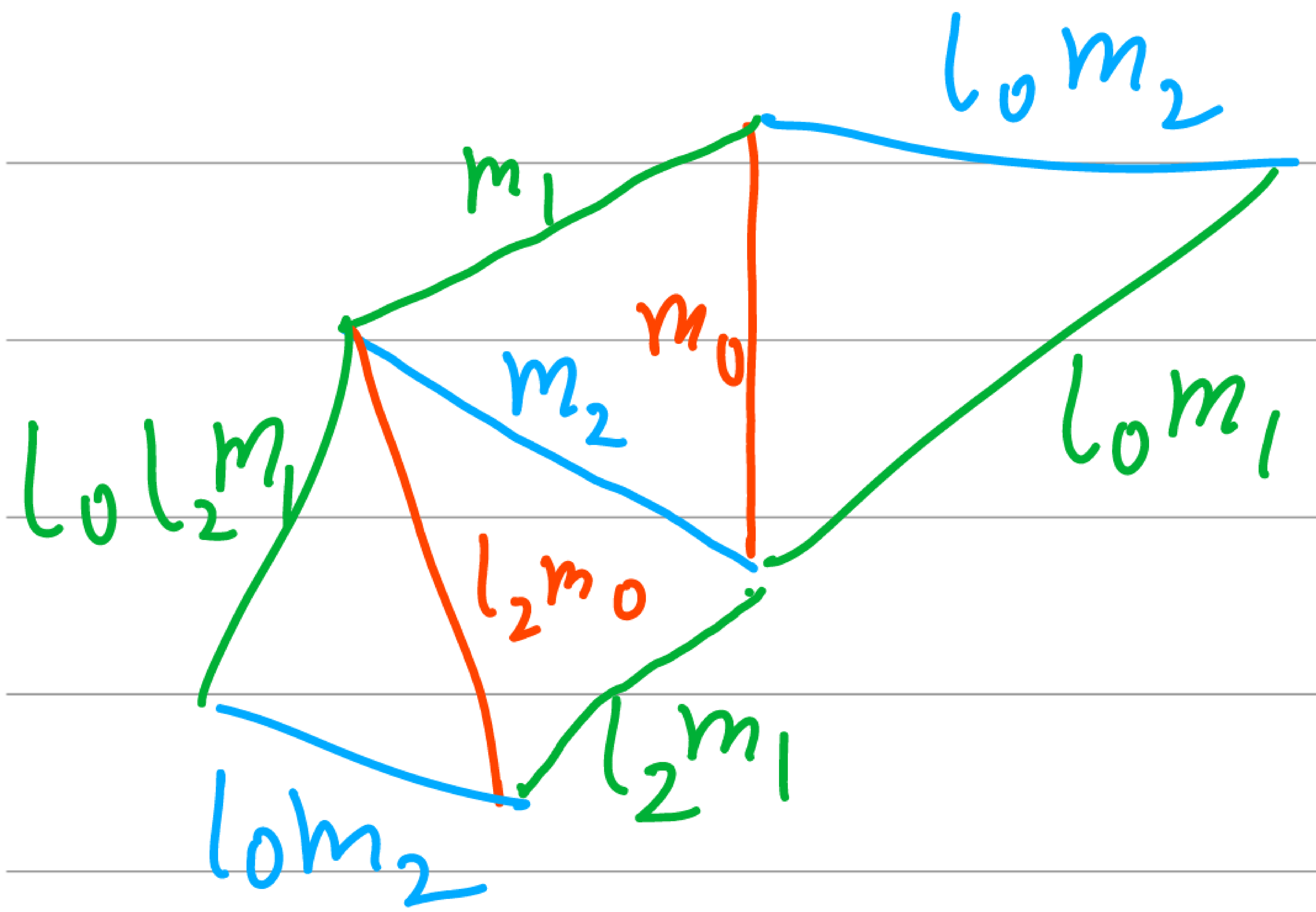
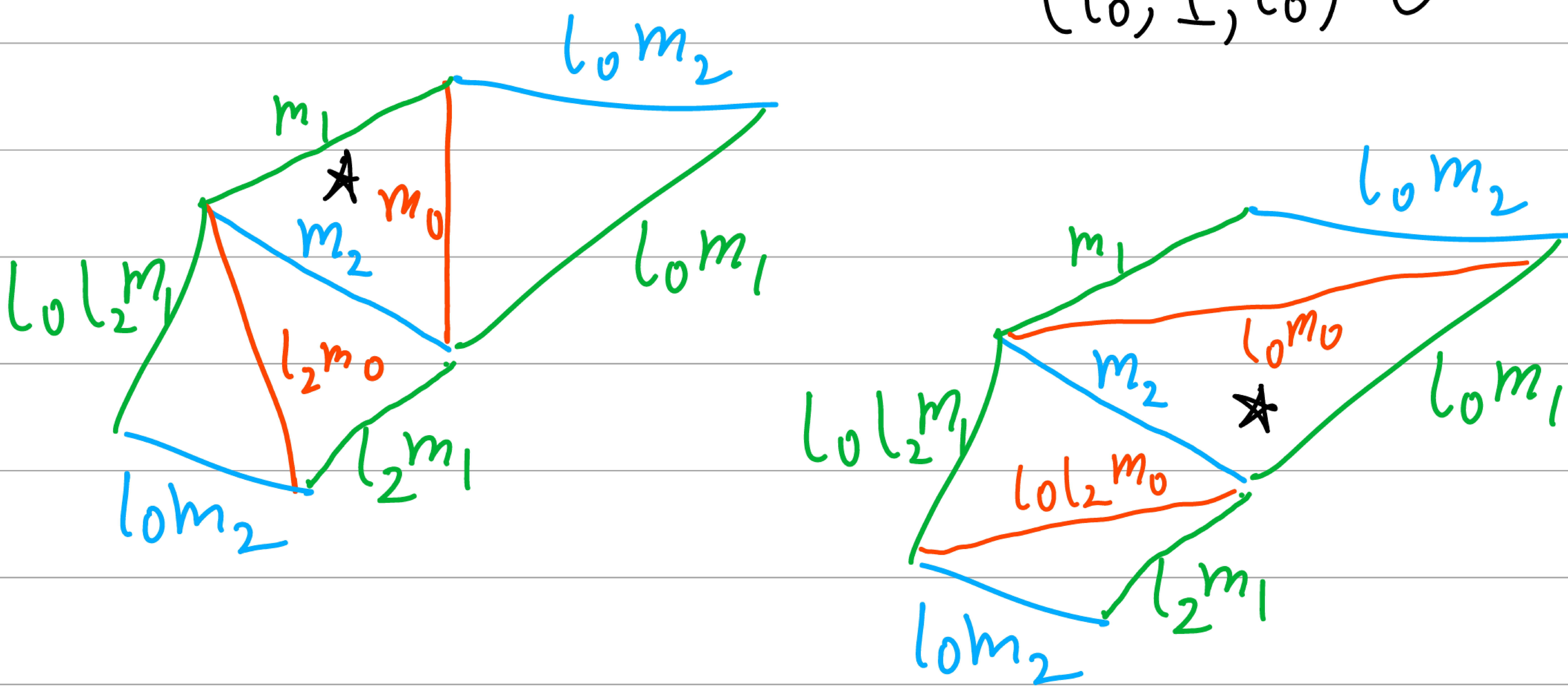
Lemma If we flip all edges with index 0, we go from $S((\iota_0, \iota_1, \iota_2), \sigma)$ to $S((\iota_0, \iota_0\iota_1\iota_0, \iota_0\iota_2\iota_0), (\iota_0, I, \iota_0) \cdot \sigma)$.

Lemma If we flip all edges with index σ, we go from $S((l_0, l_1, l_2), \sigma)$ to $S((l_0, l_0 l_1 l_0, l_0 l_2 l_0), (l_0, l_0, I) \cdot \sigma)$.



Lemma If we flip all edges with index 0, we go from $S((l_0, l_1, l_2), \mathcal{O})$ to $S((l_0, l_0 l_1 l_0, l_0 l_2 l_0), (l_0, l_0, I) \cdot \mathcal{O})$.

$$(l_0, l_0, I) \cdot \mathcal{O} = (l_0, I, l_0) \cdot \mathcal{O}$$



Lemma If we flip all edges with index $\mathcal{O}$, we go from $S = S((\iota_0, \iota_1, \iota_2), \mathcal{O})$ to $S_0^\circ = S((\iota_0, \iota_0\iota_1\iota_0, \iota_0\iota_2\iota_0), (\iota_0, \iota_0, I)\cdot\mathcal{O})$.

Thus, $\iota_0 \cdot S = \iota_0 \cdot S_0^\circ = S(\vec{\iota}, (I, I, \iota_0)\cdot\mathcal{O})$

Lemma If we flip all edges with index 0, we go from $S = S\big((\iota_0, \iota_1, \iota_2), \theta\big)$ to $S_0' = S\big((\iota_0, \iota_0\iota_1\iota_0, \iota_0\iota_2\iota_0), (\iota_0, \iota_0, I)\cdot\theta\big)$.

Thus,

$$\iota_0 \cdot S = \iota_0 \cdot S_0' = S\big(\vec{\iota}, (I, I, \iota_0)\cdot\theta\big)$$

$$\iota_1 \cdot S = \iota_1 \cdot S_1' = S\big(\vec{\iota}, (\iota_1, I, I)\cdot\theta\big)$$

$$\iota_2 \cdot S = \iota_2 \cdot S_2' = S\big(\vec{\iota}, (I, \iota_2, I)\cdot\theta\big)$$

Lemma If we flip all edges with index $\mathcal{O}$, we go from $S = S\big((\iota_0, \iota_1, \iota_2), \mathcal{O}\big)$ to $S_0' = S\big((\iota_0, \iota_0\iota_1\iota_0, \iota_0\iota_2\iota_0), (\iota_0, \iota_0, I)\cdot\mathcal{O}\big)$.

Thus,

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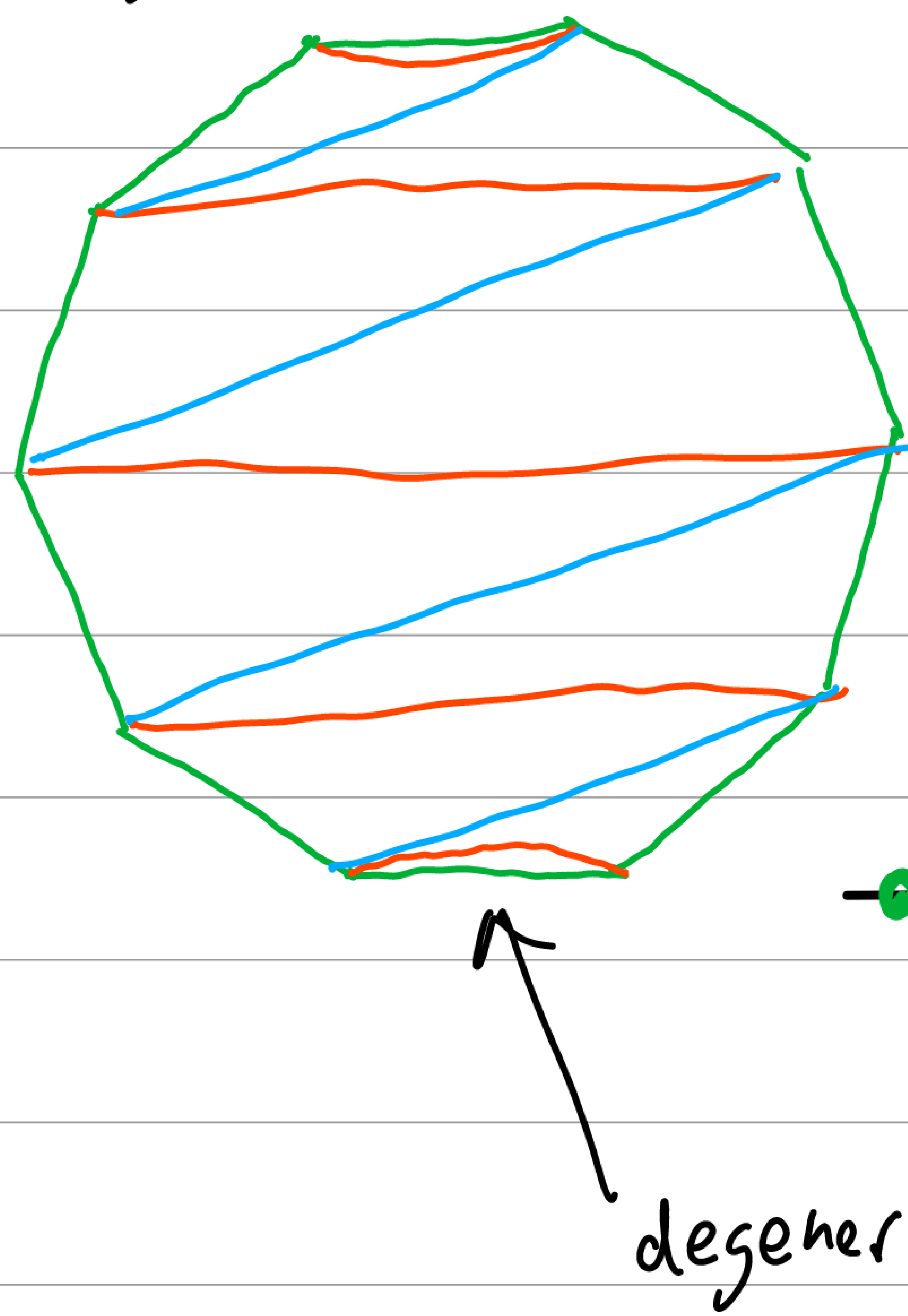
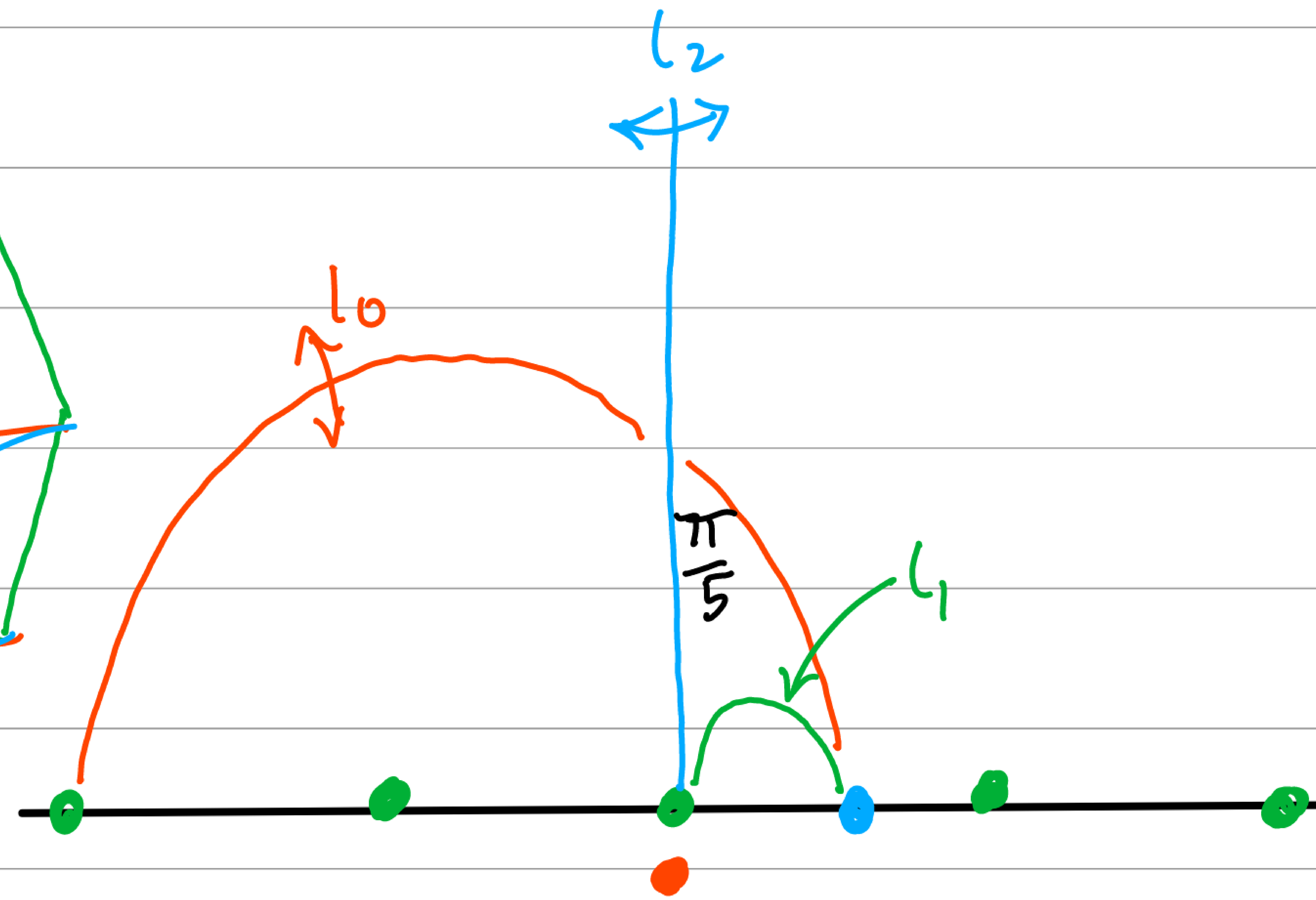
$$\iota_1 \cdot S = \iota_1 \cdot S_1' = S\big(\vec{\iota}, (\iota_1, I, I)\cdot\mathcal{O}\big)$$

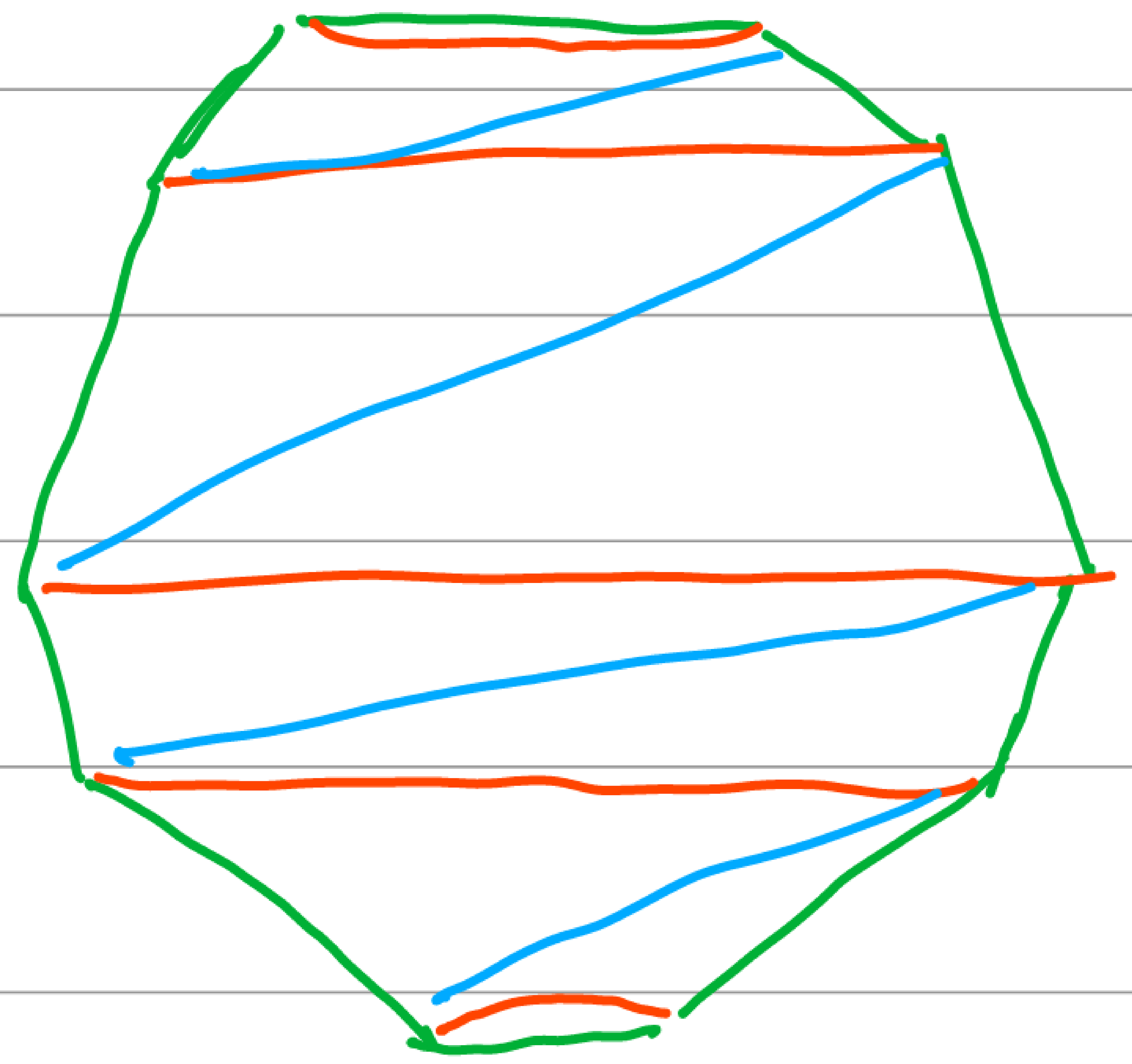
$$\iota_2 \cdot S = \iota_2 \cdot S_2' = S\big(\vec{\iota}, \underbrace{(I, \iota_2, I)}_{(\mathbb{Z}/2\mathbb{Z})^3\text{-action}}\cdot\mathcal{O}\big)$$

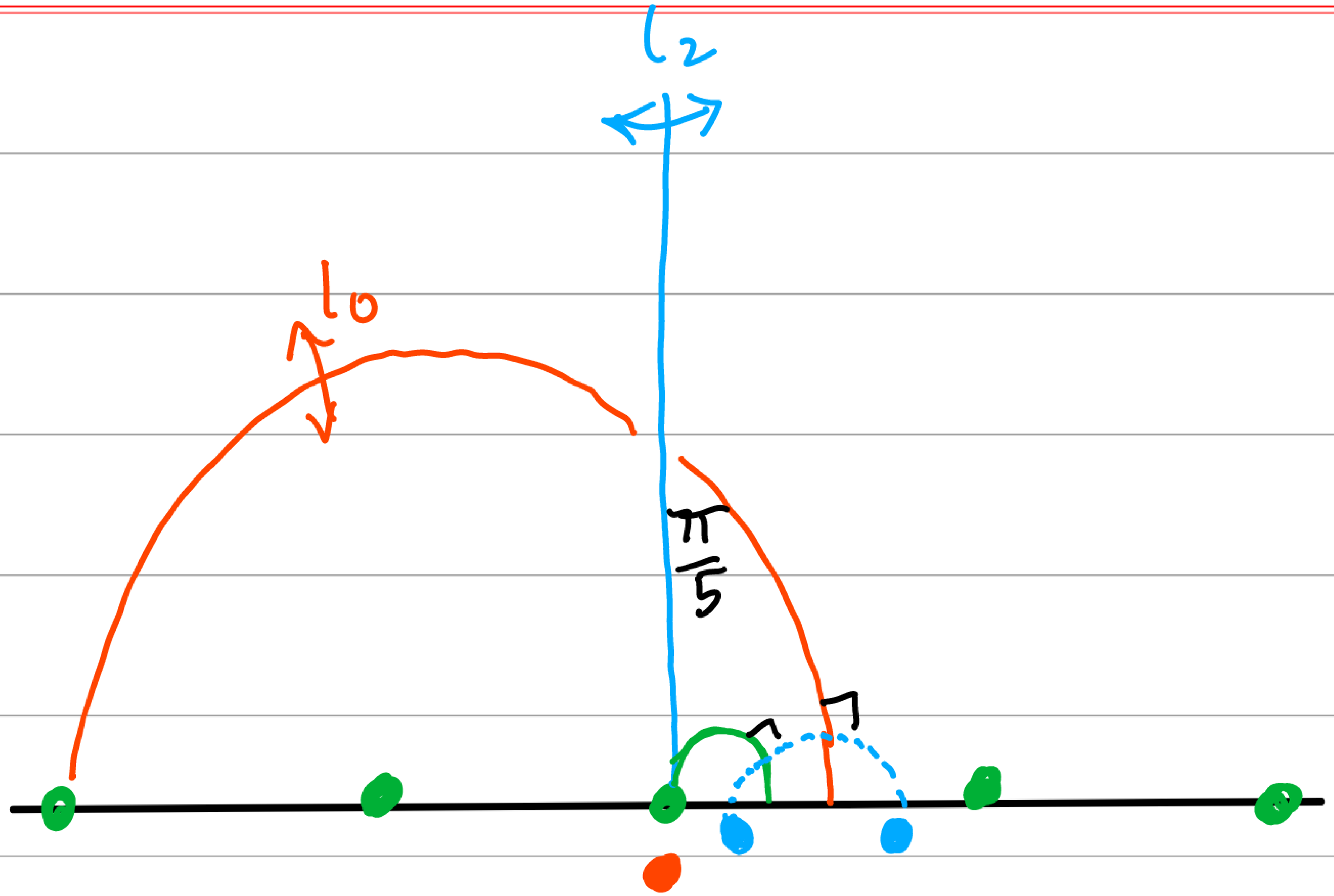
Ex Deforming Veech's Surfaces.

Regular 10-gon

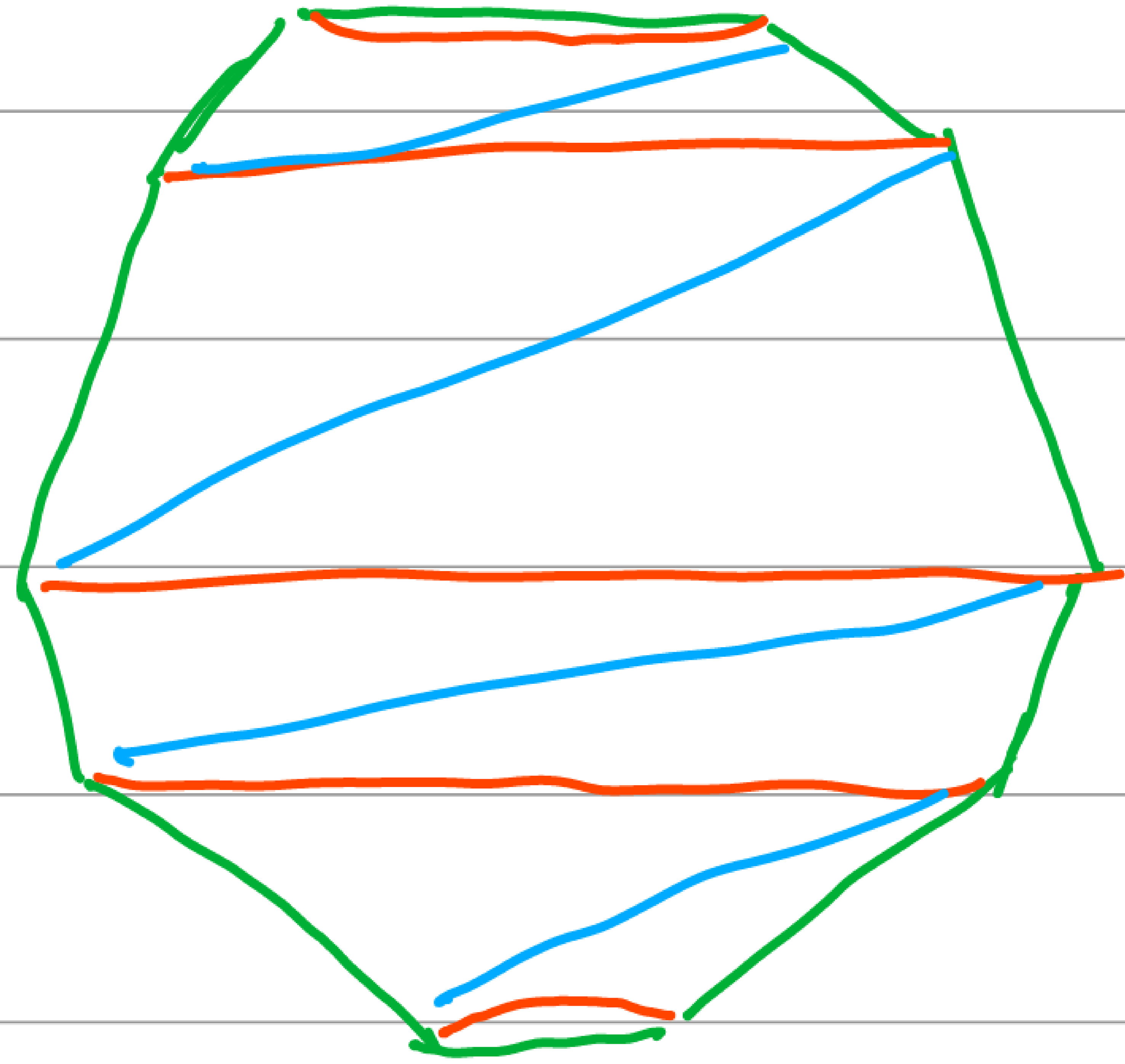


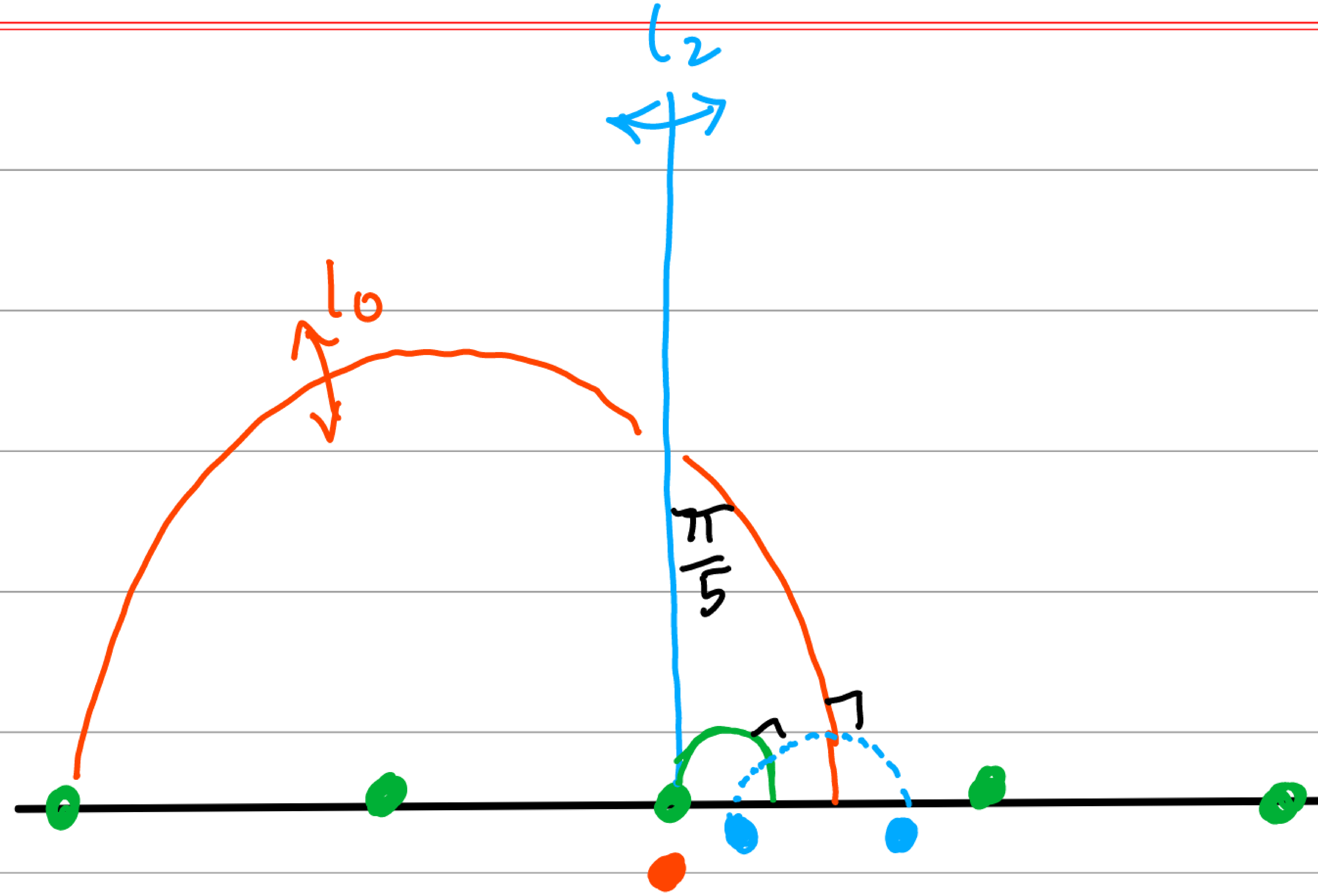






l_2
l_0
$\frac{\pi}{5}$





Infinite variant