Geodesic representatives on surfaces without metrics

arXiv:2301.03727

RTG Colloquium, Heidelberg

January 16, 2024

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joint work with Ferrán Valdez and Barak Weiss.

Talk Outline

1) Surfaces from polygons: Translation + Dilation Surfaces

2) Tori
   ⓐ Translation structures
   ⓑ Dilation structures
   ⓒ Realization theorem for homotopy classes of curves
   ⓓ Zebra structures

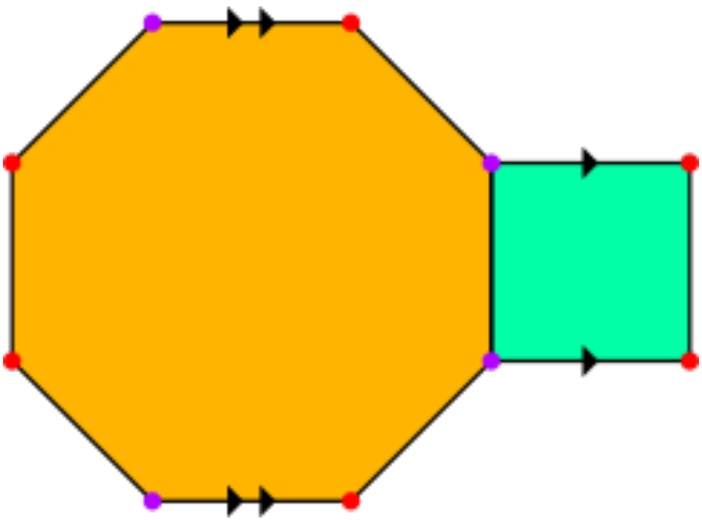
3) Higher genus surfaces

Surfaces from Euclidean polygons

We will construct surfaces from polygons with edges glued in pairs. We use a group $G < \mathrm{Homeo}_+(\mathbb{R}^2)$ to identify edges.

$G = \{\text{translations}\}$

Translation Surface



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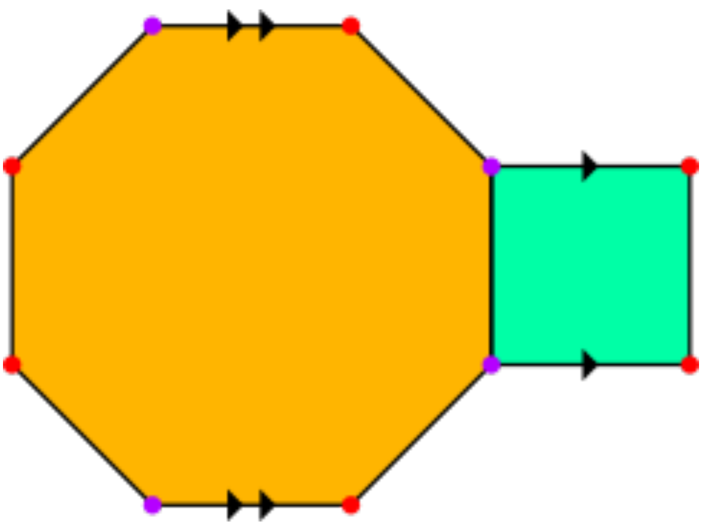
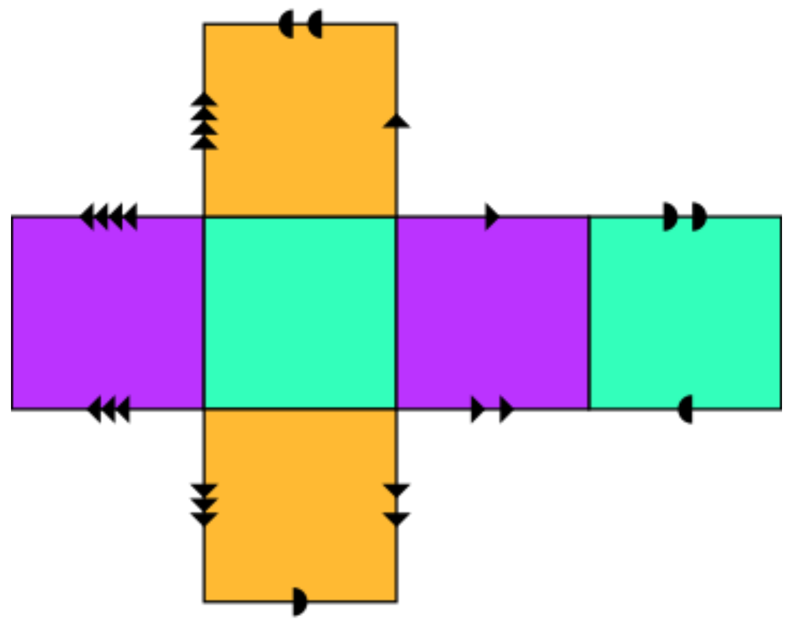
$G = \{\text{translations}\}$
Translation Surface

$G = \mathrm{Isom}_+ \mathbb{R}^2$
Euclidean cone surface.





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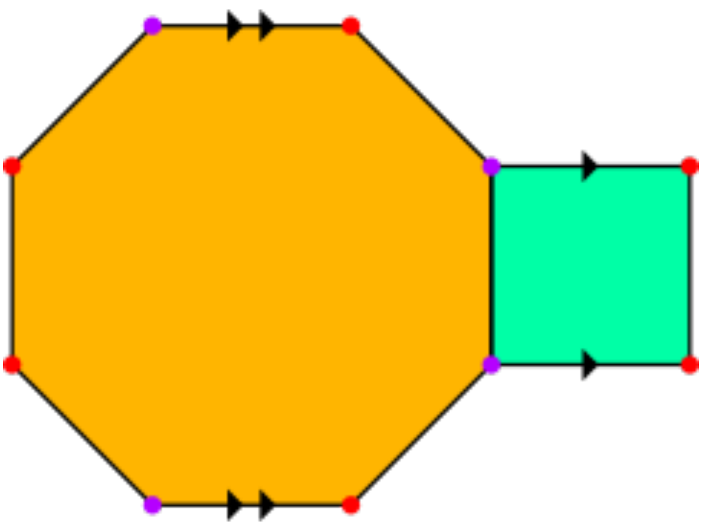
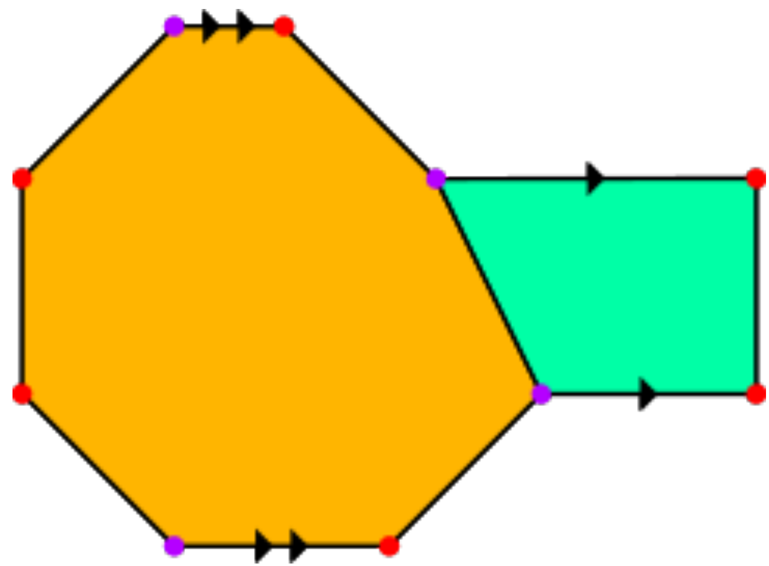


$G = \langle \text{Dilations}, \text{Translations} \rangle$
Dilation Surface



Topics studied related to Dilation Surfaces:

- Algebraic structure of moduli spaces (Veech, Apisa - Bainbridge - Wang)
- Affine symmetry groups (Duryev - Fougeron - Ghazouani)
- Affine realization of mapping classes (Wang)
- Dynamics of directional foliations (Liousse, Bowman - Sanderson, Boulanger - Fougeron - Ghazouani)
- Existence of closed leaves (Boulanger - Ghazouani - Tahar)

Related ideas:

- Affine interval exchange maps (Camelier - Gutierrez, Cobo, Cobo - Gutiérrez-Romo - Maass, Marmi - Moussa - Yoccoz, ...)
- Twisted measured laminations (McMullen, for studying fibered 3-manifolds)
- Infinite translation surfaces (Hooper - Hubert - Weiss)

Goal of this talk:

To understand the geometry of homotopy classes of closed curves

Are there canonical representatives?

What properties do they have?

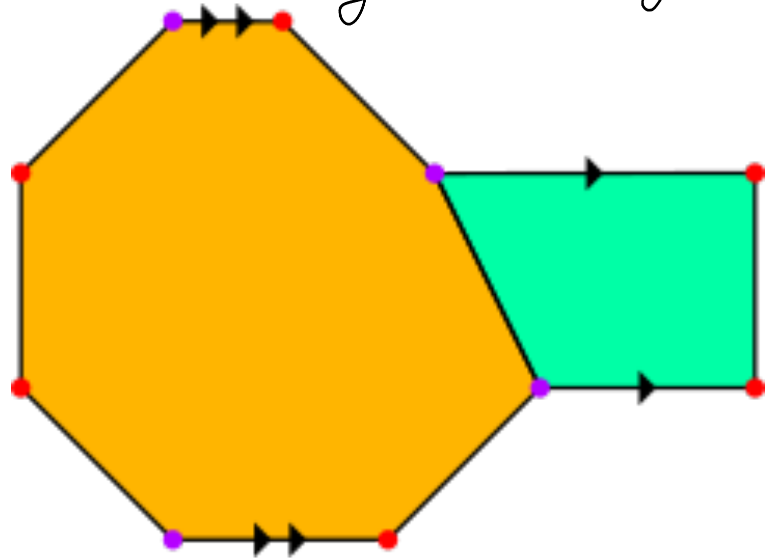
Gauss–Bonnet

Points formed from identified vertices form the singularities of our surface. The angle of a singularity p is Θ_p, the sum of interior angles at vertices identified to create p.

Theorem Let S be a surface formed by identifying polygons and let g be its genus.

Then

$$\sum_{p \in \mathrm{sing}(S)} \Theta_p - 2\pi = 4\pi(g-1).$$



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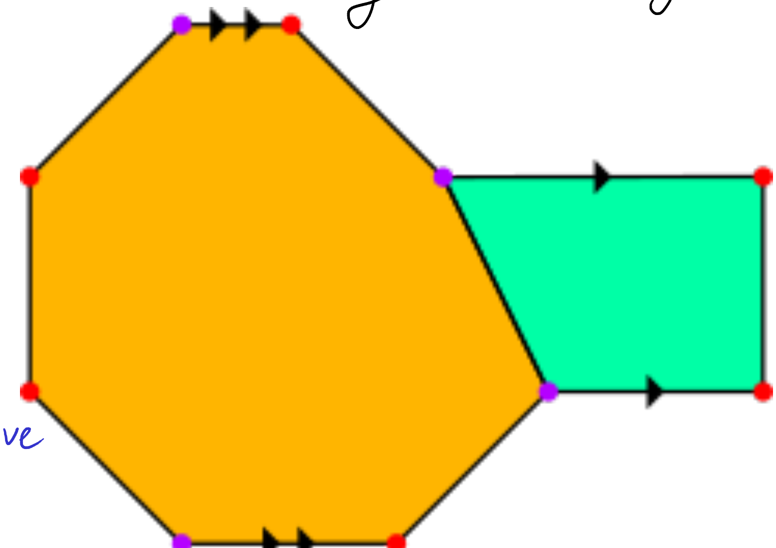


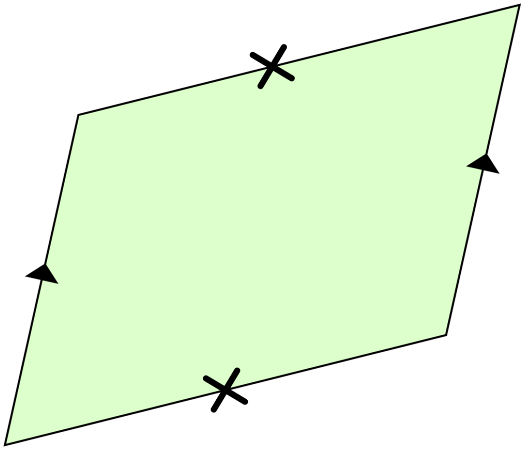
Example Both the red and purple singularities have $\Theta_p = 4\pi$ so $g = 2$.

Tori

Pic from youtube video "Homer Simpson Donuts recipe"
https://www.youtube.com/watch?v=MqXPADrPc94

Translation surface tori

Let S be a torus with a translation structure. Then the universal cover is $\mathbb{R}^2$ and $S = \mathbb{R}^2/\Lambda$ where $\Lambda < \mathbb{R}^2$ is a lattice in the translation group.



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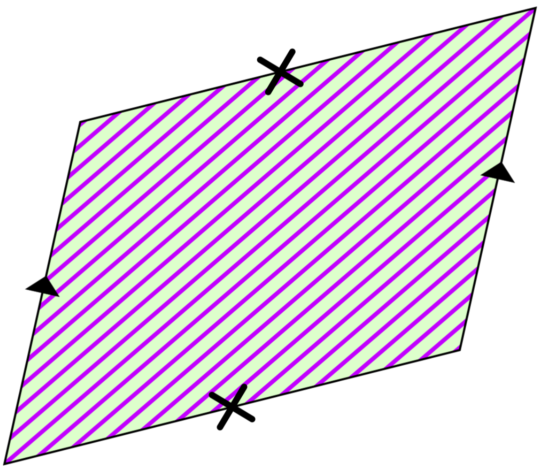
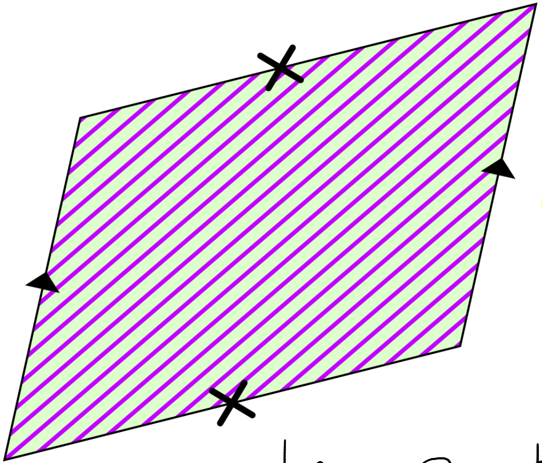


For each slope $m \in \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ there is a directional foliation $\mathcal{F}_m$ by lines of slope m.

Translation surface tori

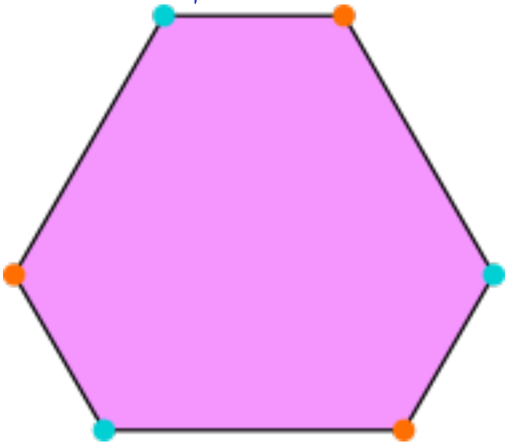


For each slope $m \in \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ there is a directional foliation $\mathcal{F}_m$ by lines of slope m.

In particular, for every homotopy class $[\gamma]$ of an essential simple closed curve γ, there is an m such that $\mathcal{F}_m$ has closed leaves in the homotopy class γ.

Dilation surface tori

Example 1:



Dilation surface tori

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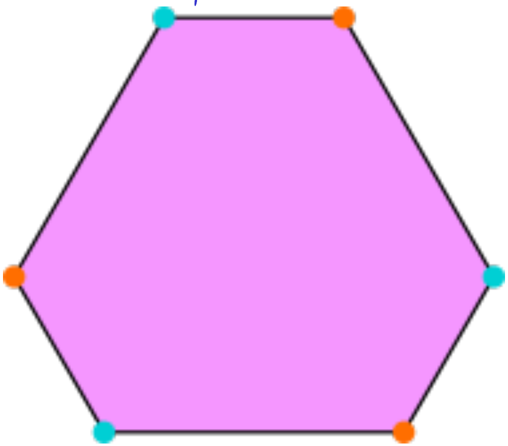
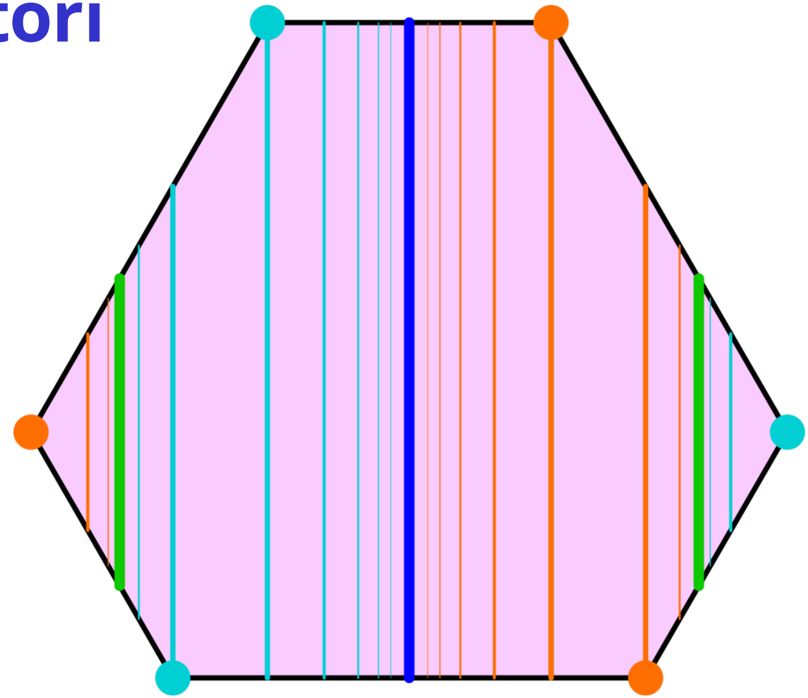
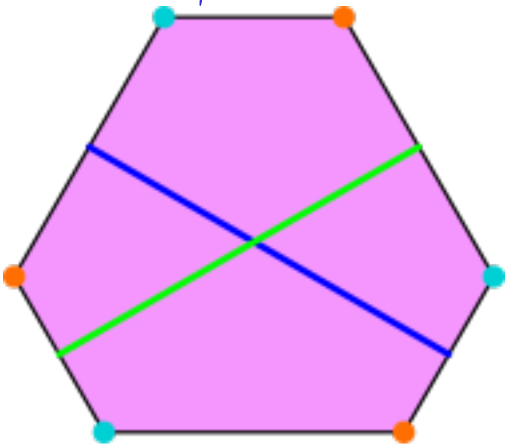




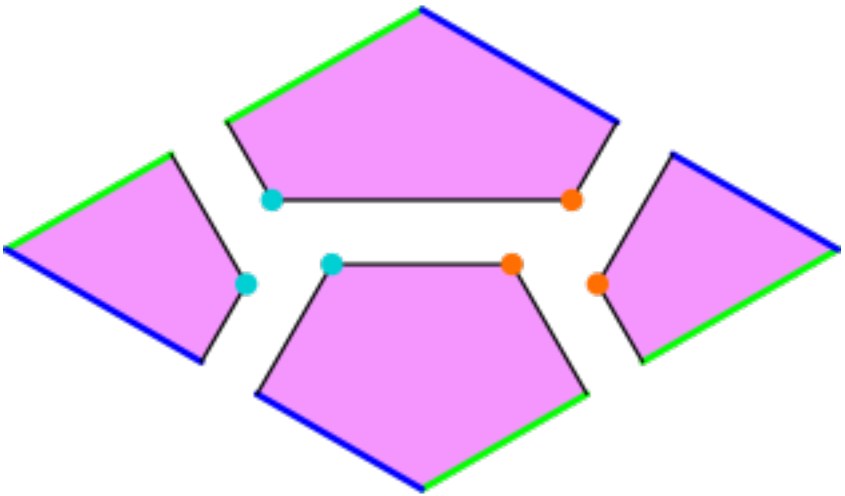
Dilation singularities distort but do not destroy the straight-line foliations.

Dilation surface tori

Example 1:



Slicing along closed leaves produces a generalized parallelogram (with singularities).



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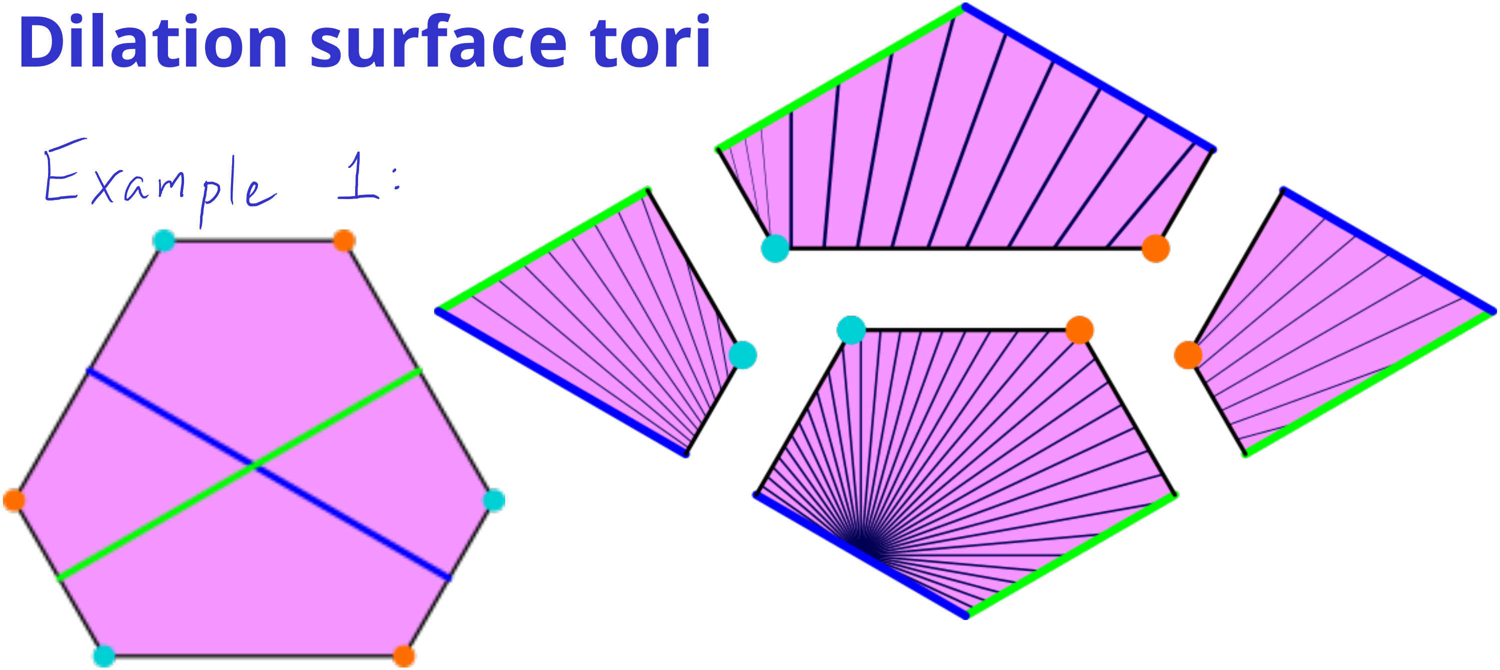

Lemma: Lines emanating from a point in the boundary of a generalized convex polygon foliate the polygon.

Dilation surface tori

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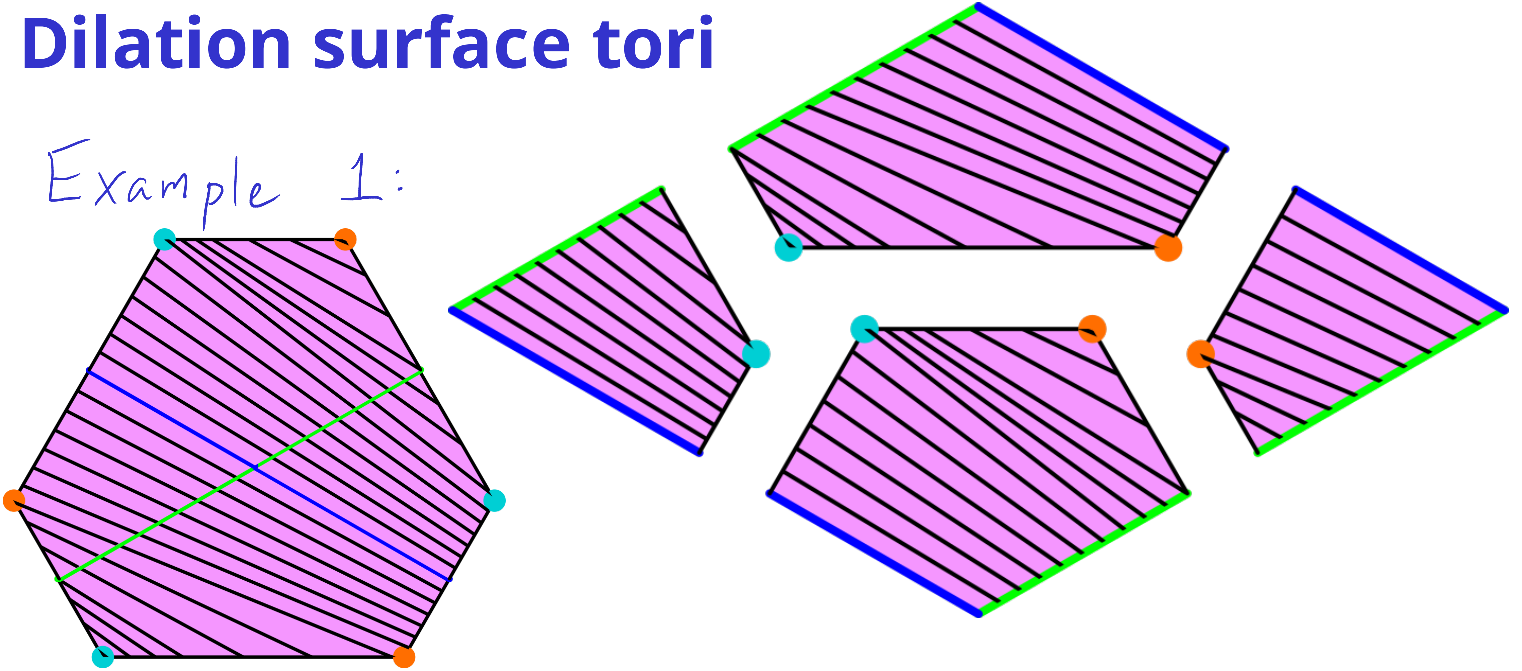

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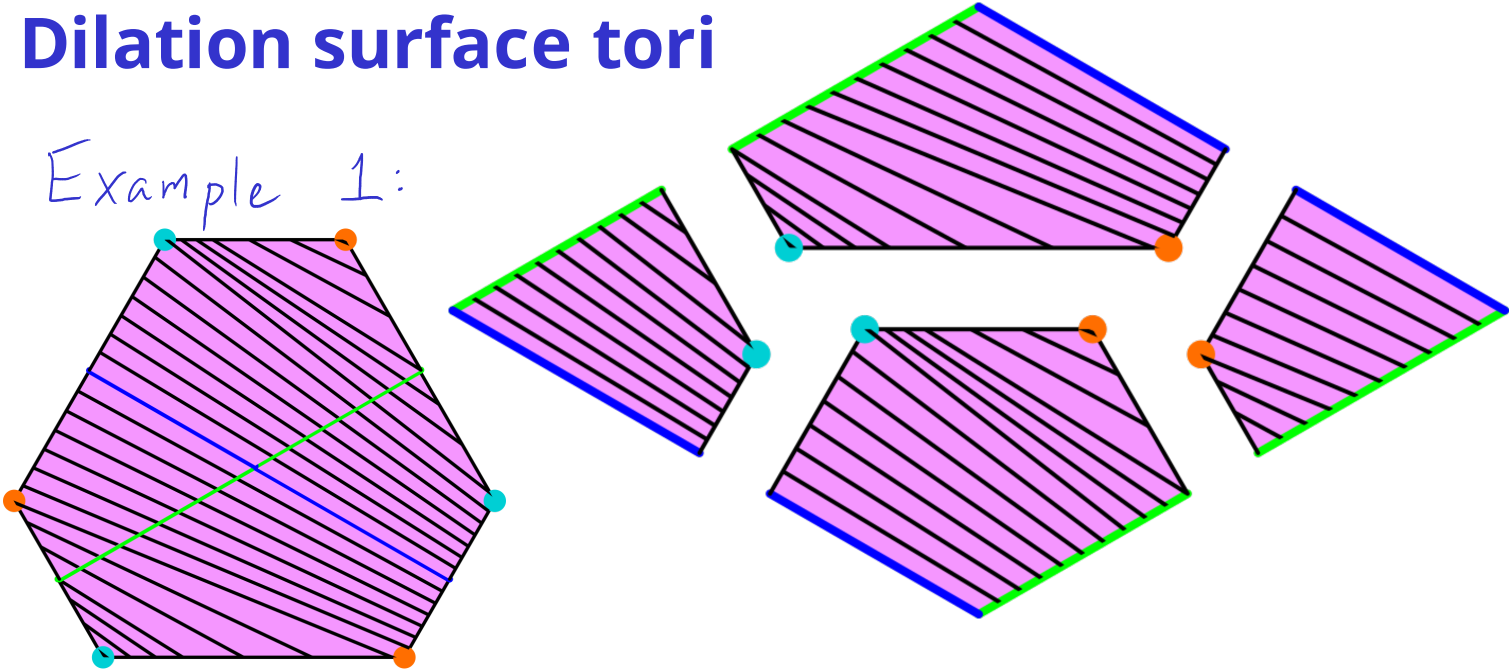
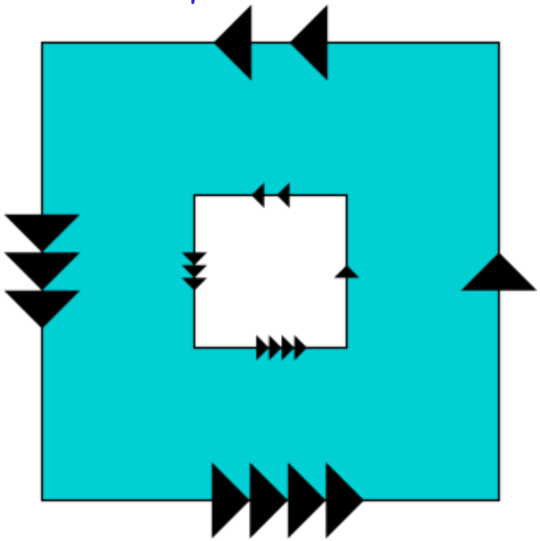
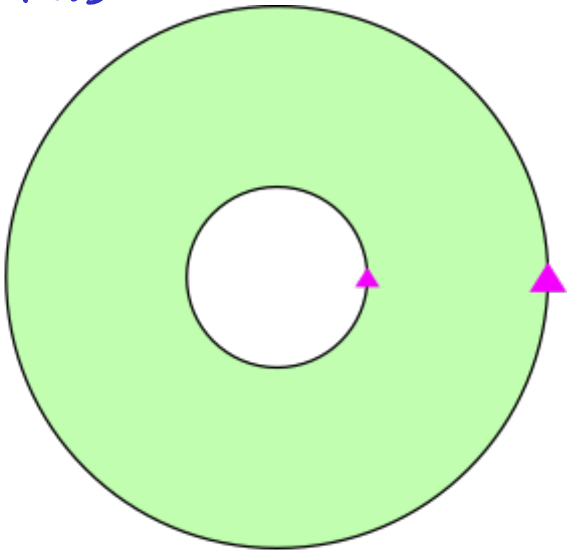


Theorem (HVW) A dilation torus with two non-homotopic closed leaves has the property that there is a foliation by closed leaves in every homotopy class of an essential simple closed curve (scc).

Dilation surface tori

Example 2: The Hopf torus

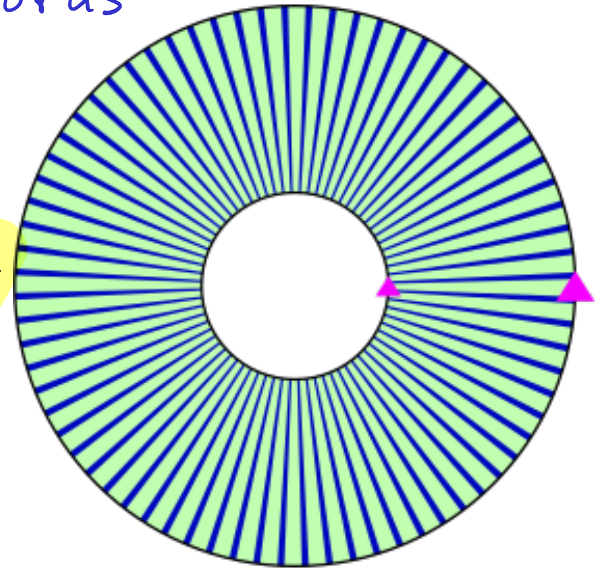




Dilation surface tori

Example 2: The Hopf torus

The Hopf torus has a foliation by closed leaves with two leaves of every slope.



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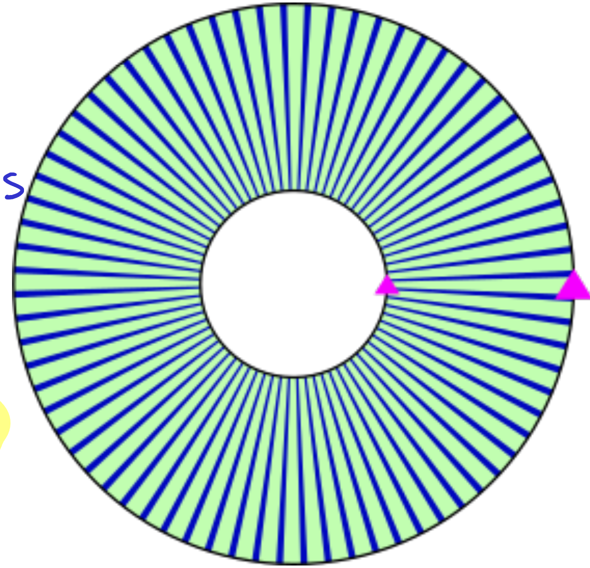
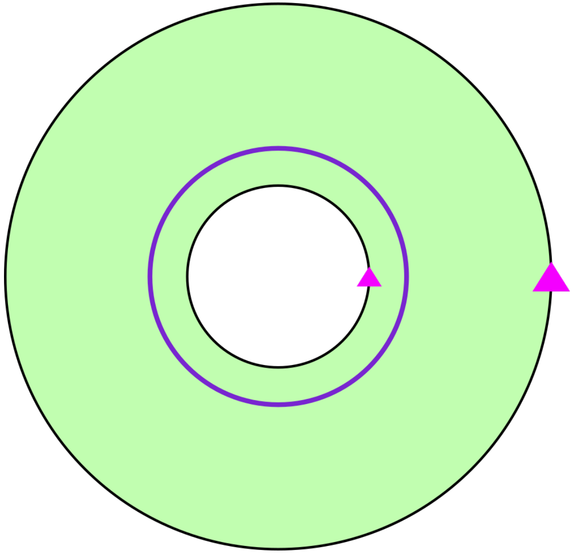


The Hopf torus has a foliation by closed leaves with two leaves of every slope.

Observe:

There is no closed leaf in the homotopy class of the purple simple closed curve.



Dilation surface tori

Theorem (HVW w/ ideas of Selim Ghazouani)

Let S be a dilation torus (possibly with 2π-dilation singularities). Then either

① For every essential simple closed curve γ, there is a foliation of S by straight lines in $[\gamma]$.

or ② There is a unique escc γ for which there is such a foliation and this foliation has leaves of all slopes.

Dilation surface tori

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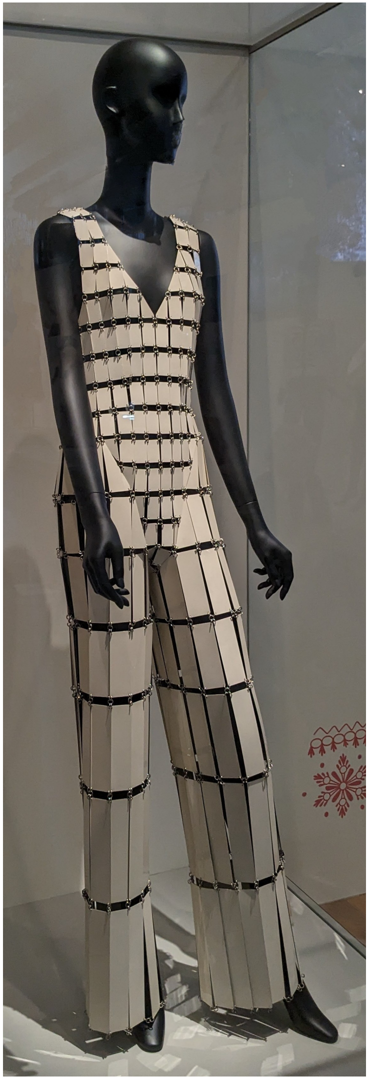
2. There is a unique escc γ for which there is such a foliation and this foliation has leaves of all slopes.

This theorem holds for "zebra" structures on tori as well...

Two guiding questions:

① What did we need for the Theorem?

② What happens as the number of dilation singularities tends to infinity?



Paco Rabanne, Combinaison, Collection haute couture, Spring 1997, as seen in Marseille

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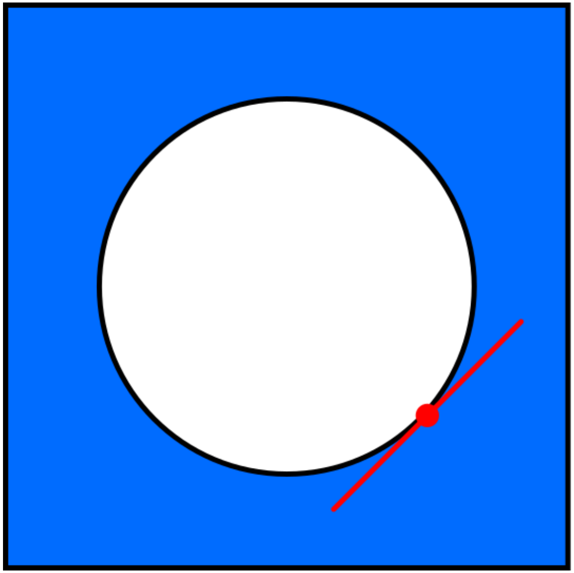

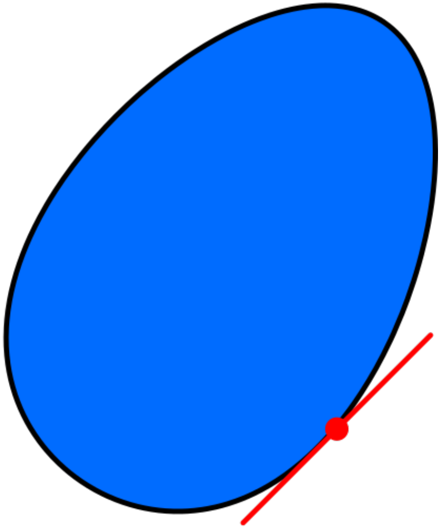

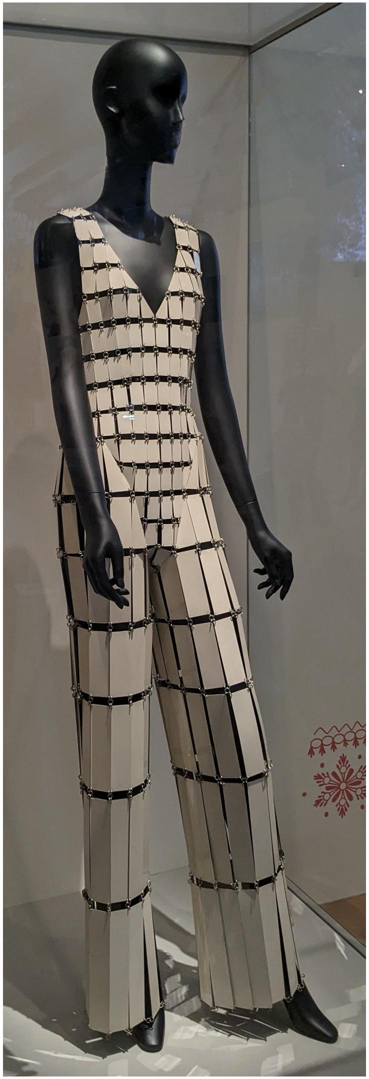

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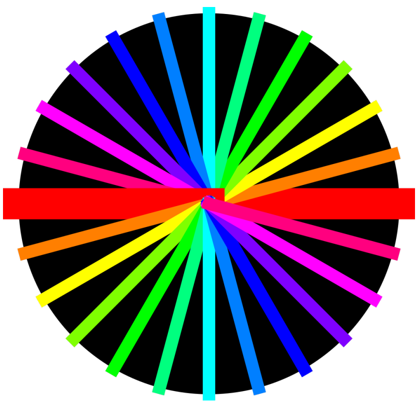
Definition of a Zebra torus:

A zebra structure on the torus T is a collection of foliations indexed by slope,

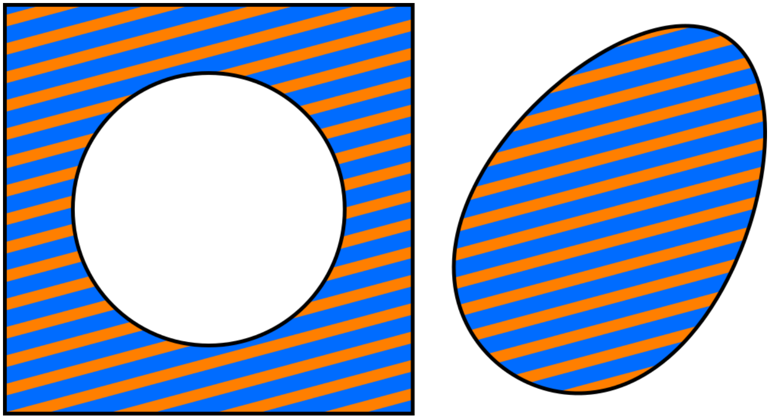
$$\{\mathcal{F}_m : m \in \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}\}$$

such that for every $p \in T$ there is a neighborhood N_p of p and a homeomorphism $h_p : N_p \to \mathbb{R}^2$ such that ① $h_p(p) = \vec{0}$, and



② $\forall m \in \hat{\mathbb{R}}$, if $\ell_m \subset N_p$ is the leaf of $\mathcal{F}_m$ through p, then $h_p(\ell_m)$ is the line of slope m through $\vec{0}$.

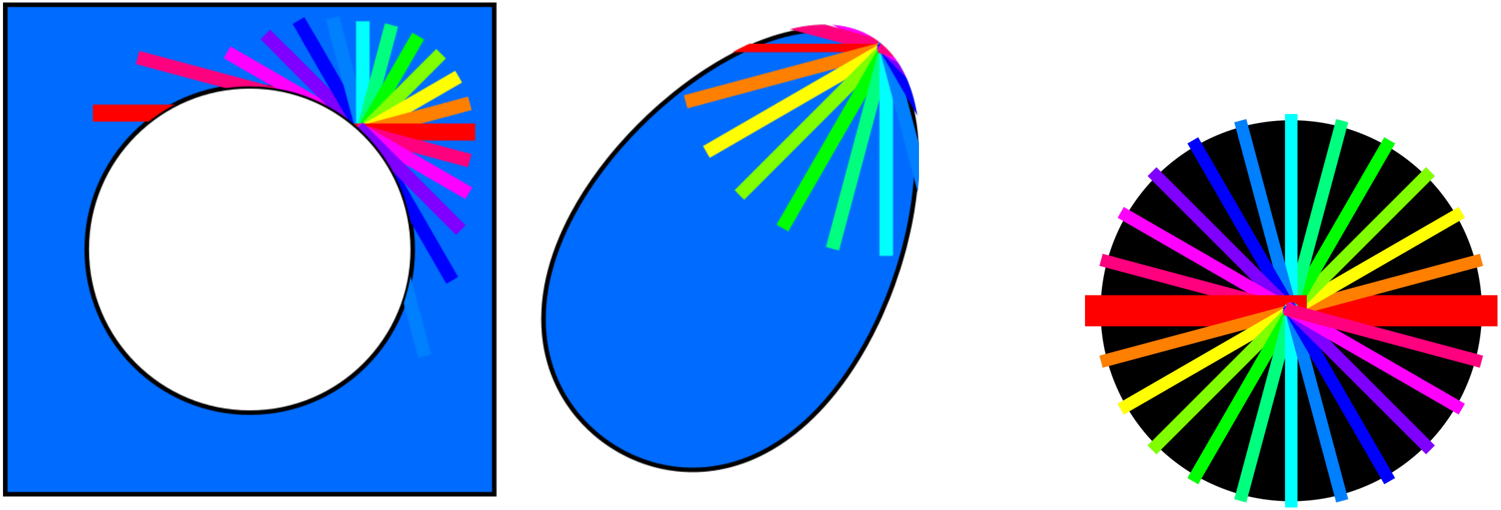
Definition of a Zebra torus:



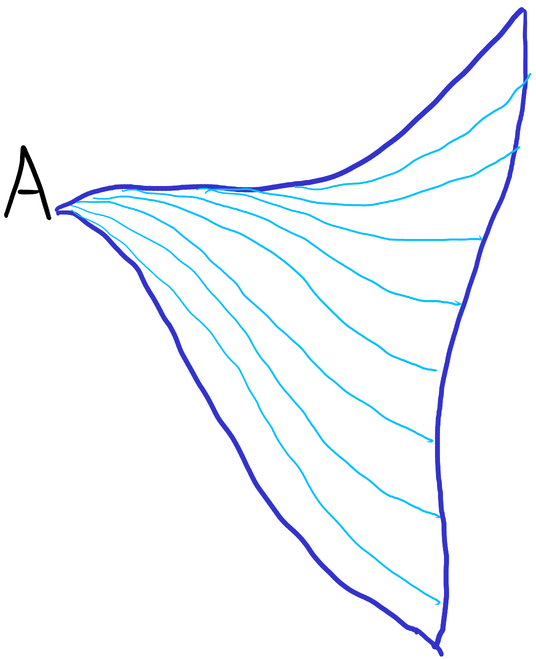
We require a foliation of every "slope"

and

pointwise local compatibility.



Important Lemma: Given a triangle ABC in a zebra surface, the leaves emanating from A foliate $\triangle ABC$.



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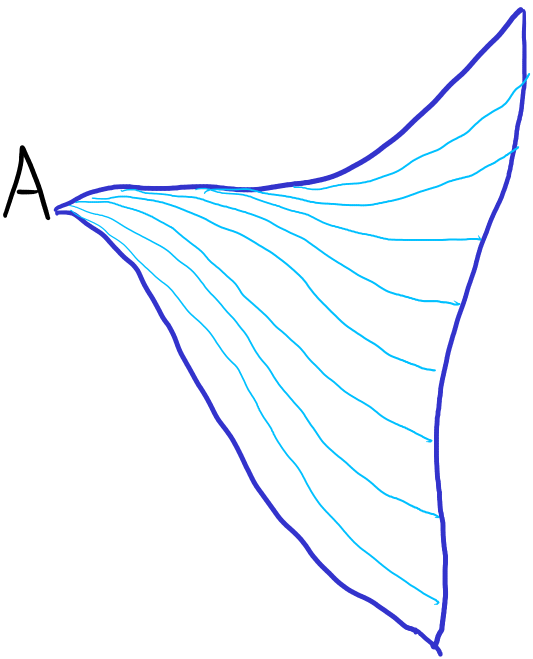
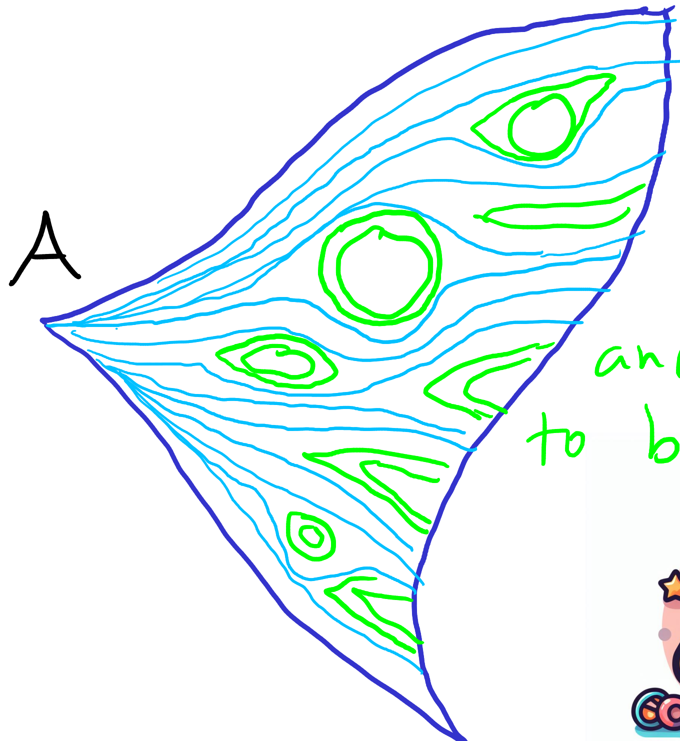
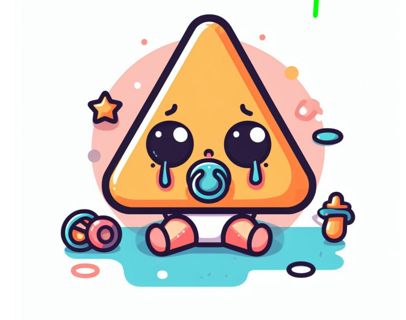
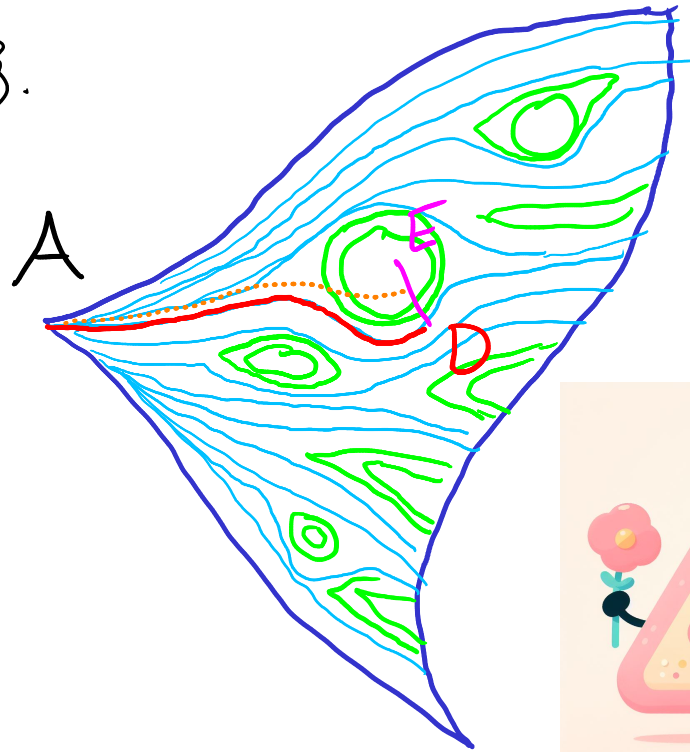




Poor triangle has gas and needs to be burped



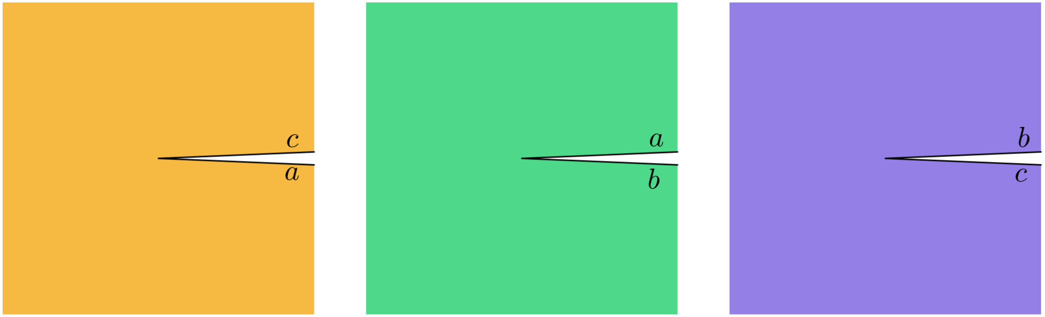
Burp lemma If $\overline{AD}$ and $\overline{DE}$ are arcs of leaves and $\measuredangle ADE < \pi$ then there is an arc of a leaf from A to a point on $\overline{DE} \setminus \{D, E\}$.



Surfaces of genus 2 and higher

Responses to the prompt "Create for me a photorealistic image of a genus two surface, covered with pink frosting and sprinkles" by Microsoft's image generator.

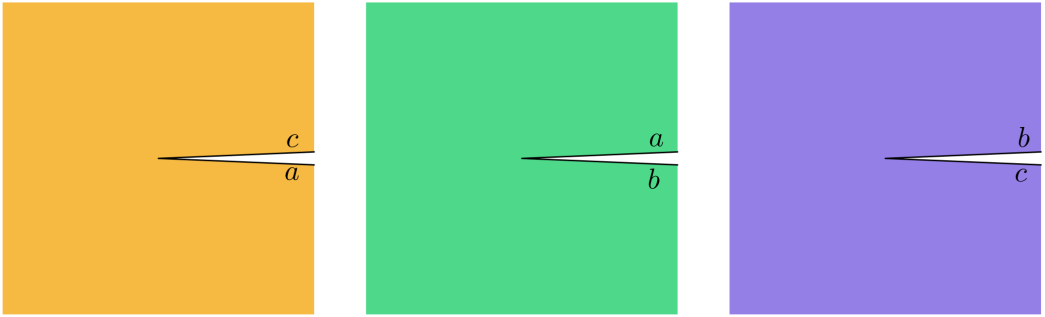
Dilation surface singularities



Model:

Slit planes $\mathbb{C} \setminus \mathbb{R}_+$ glued cyclically by dilations along boundary rays.

Dilation surface singularities



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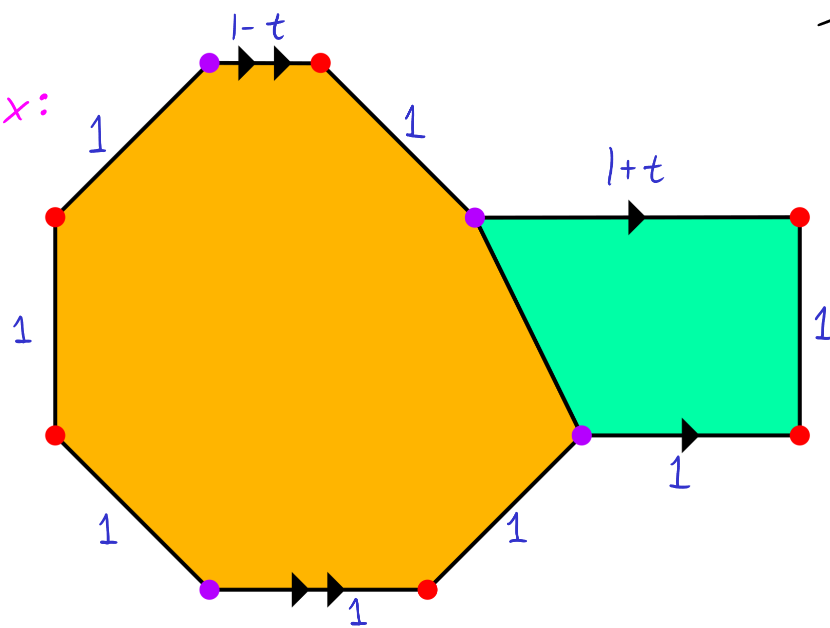
Slit planes $\mathbb{C} \setminus \mathbb{R}_+$ glued cyclically by dilations along boundary rays.

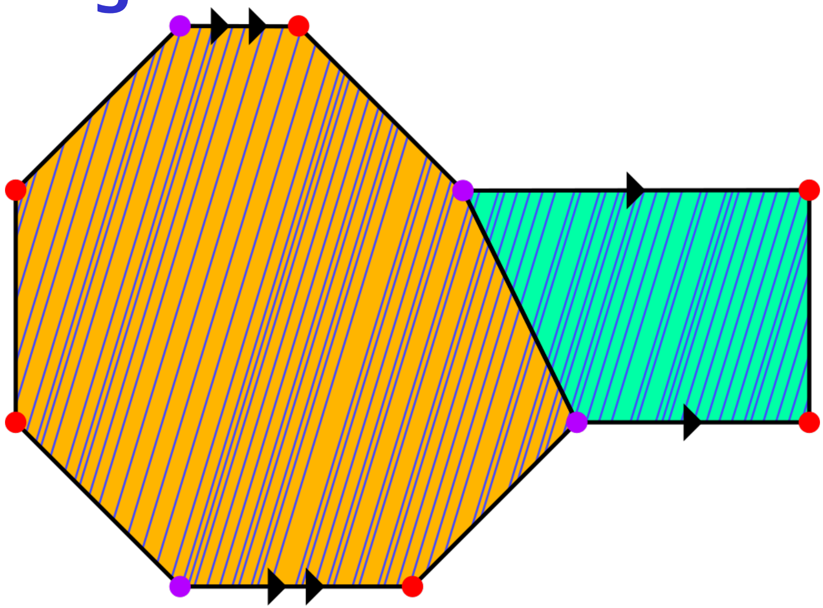
Ex:



The red singularity has angle 4π and (counterclockwise) dilation $(1+t)(1-t) = 1-t^2$.

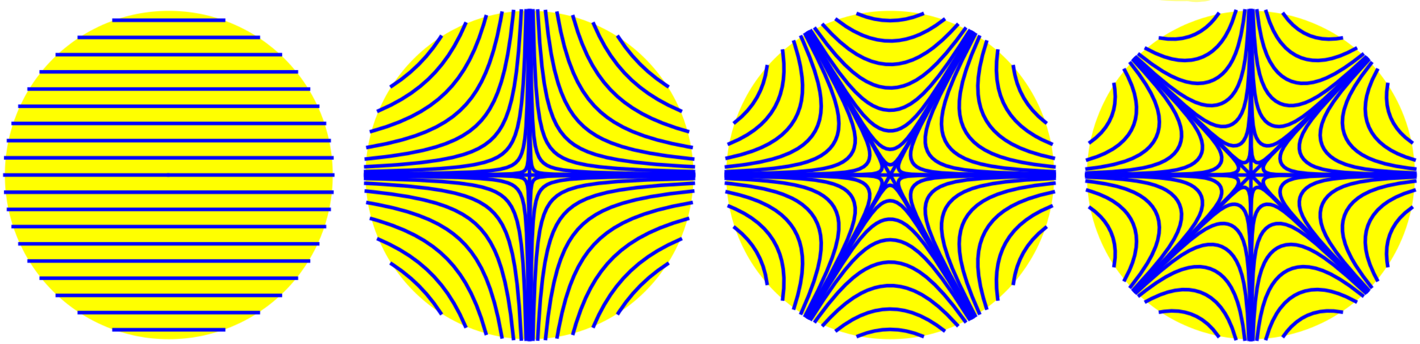
The purple singularity has angle 4π and dilation $\frac{1}{1-t^2}$.

Singular foliations



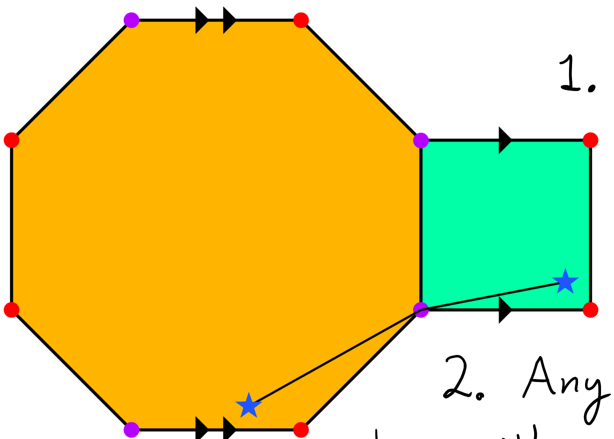
Both translation and dilation surfaces have directional foliations of every slope.

Local models for an orientable singular foliation:



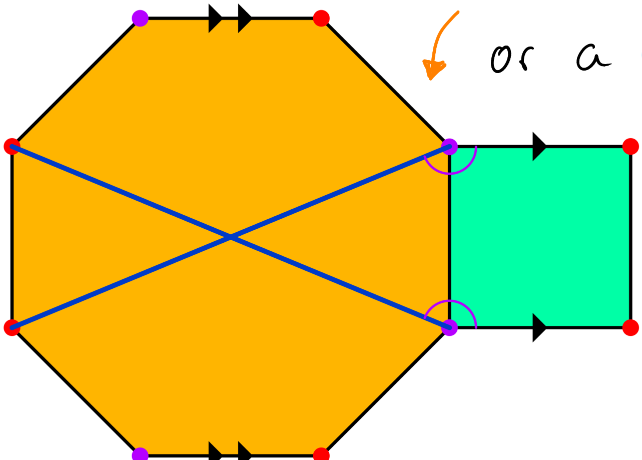
Metric geodesics in translation surfaces

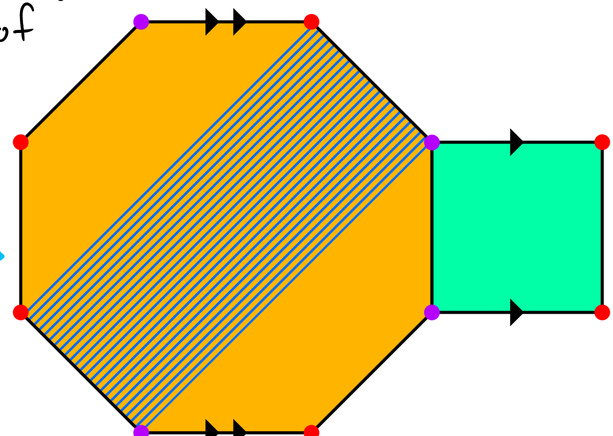
In a closed translation surface...



1. Any homotopy class of arcs rel endpoints has a metric geodesic representative.

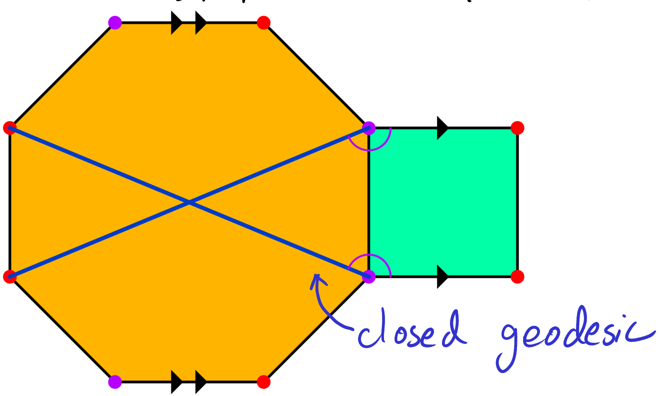
2. Any non-trivial free homotopy class of loops has either a unique bent geodesic representative or a cylinder of closed leaves.

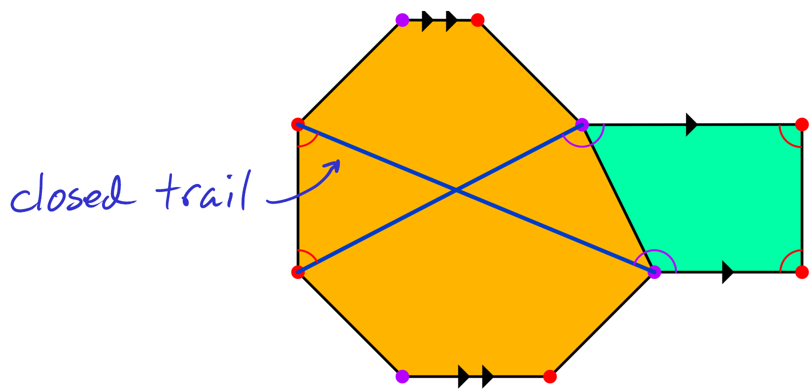




Trails in dilation surfaces

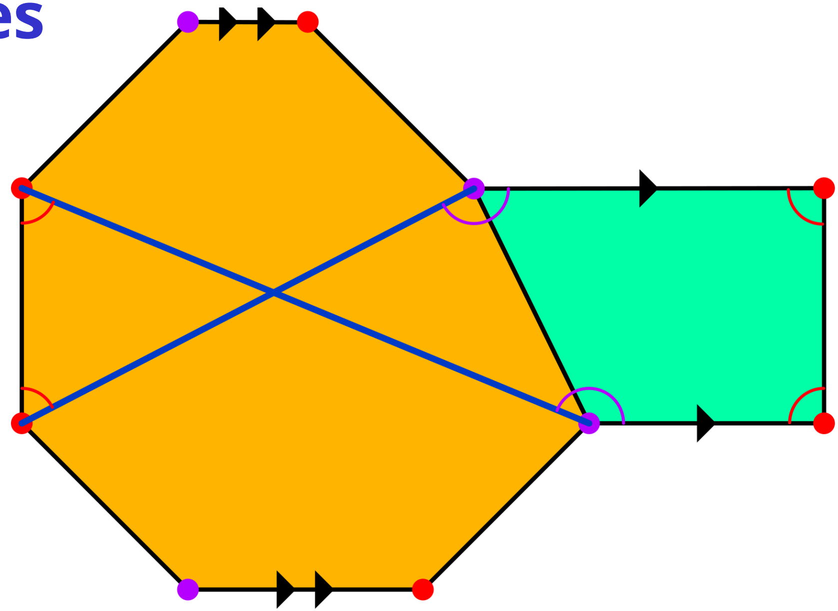
A trail in a dilation surface is a maximal bi-infinite path that follows leaves (maximal line segments), transitioning between leaves only at singularities in such a way so that the two angles made at the singular transitions measure at least π.

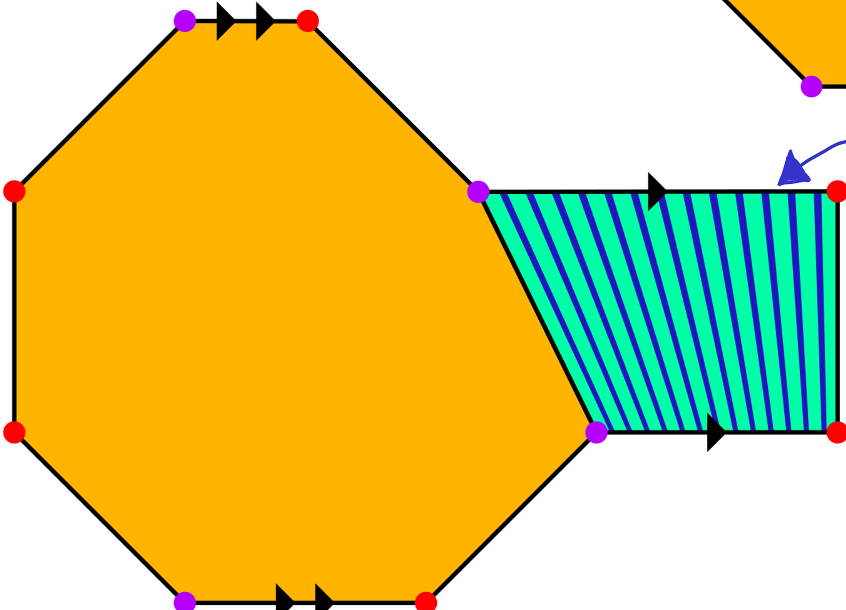




Trail Representatives

A homotopy class of closed curves represented by a unique trail.





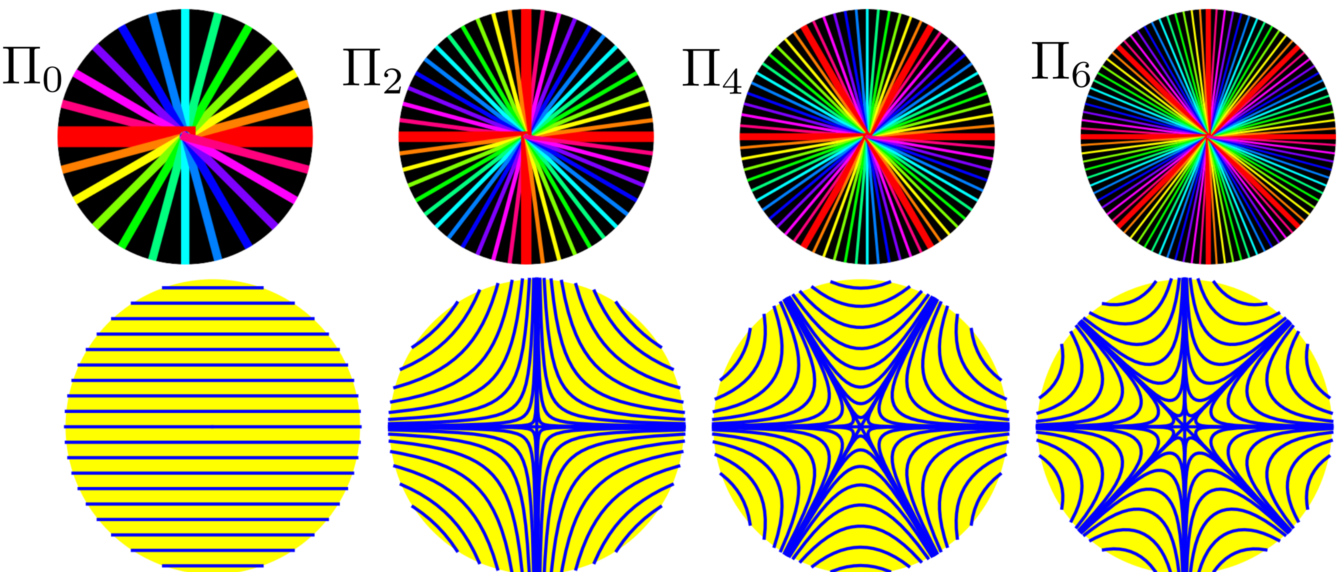
A dilation cylinder representing a homotopy class of closed curves.

Stellated foliation/zebra structures

Let S be an oriented topological surface and let $\{\mathcal{F}_m : \ m \in \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}\}$ be a collection of singular foliations indexed by slope.

We say that $\{\mathcal{F}_m\}$ is a zebra structure if:

> For each point p in S, there is an open neighborhood N containing p and a homeomorphism from N to a model space Π_k such that for all $m \in \hat{\mathbb{R}}$, the homeomorphism induces a bijection between the prongs of $\mathcal{F}_m$ at p and the rays of slope m in Π_k.



Examples of zebra structures

- Translation and dilation structures give rise to zebra structures.

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- Surfaces formed by gluing parallel edges of a union of polygons together by homeomorphism.

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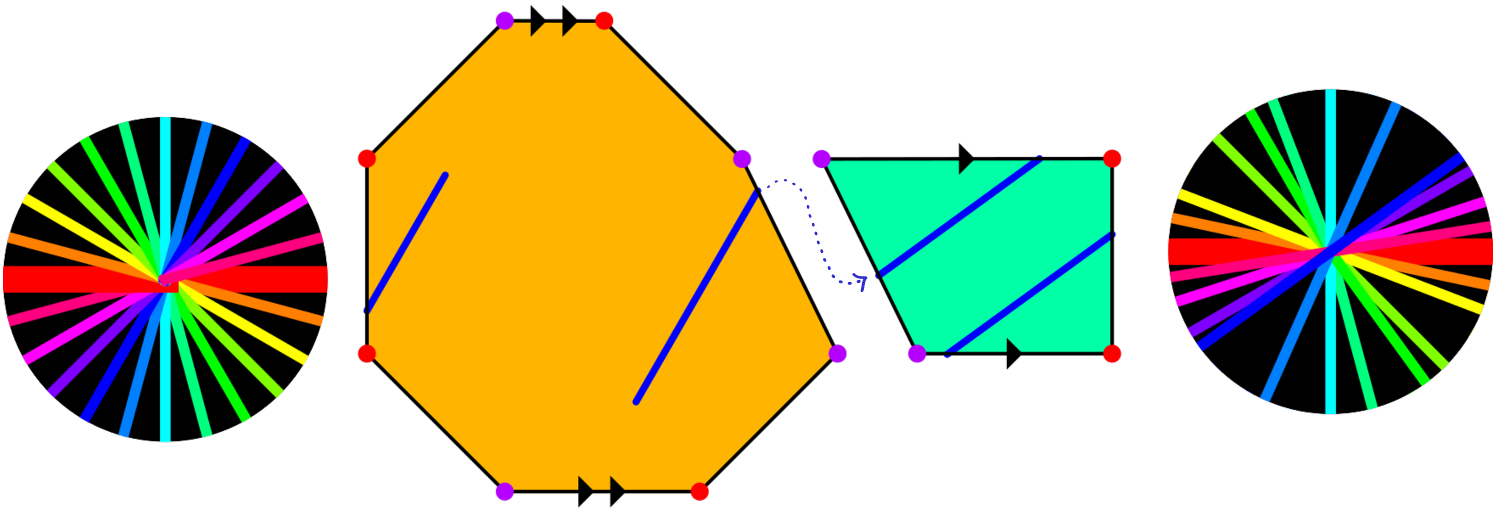
- Surfaces formed by gluing parallel edges of a union of polygons together by homeomorphism.
- $\mathrm{Homeo}_+(\hat{\mathbb{R}})$ acts on zebra structures. You can act on the polygons before gluing.



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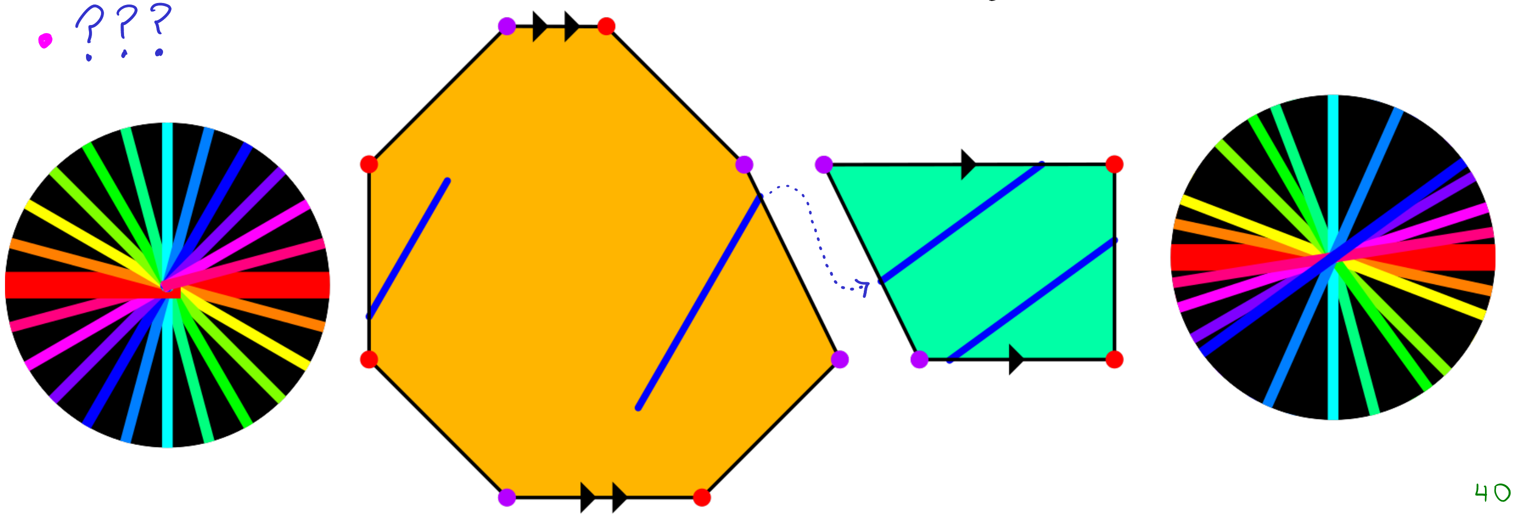
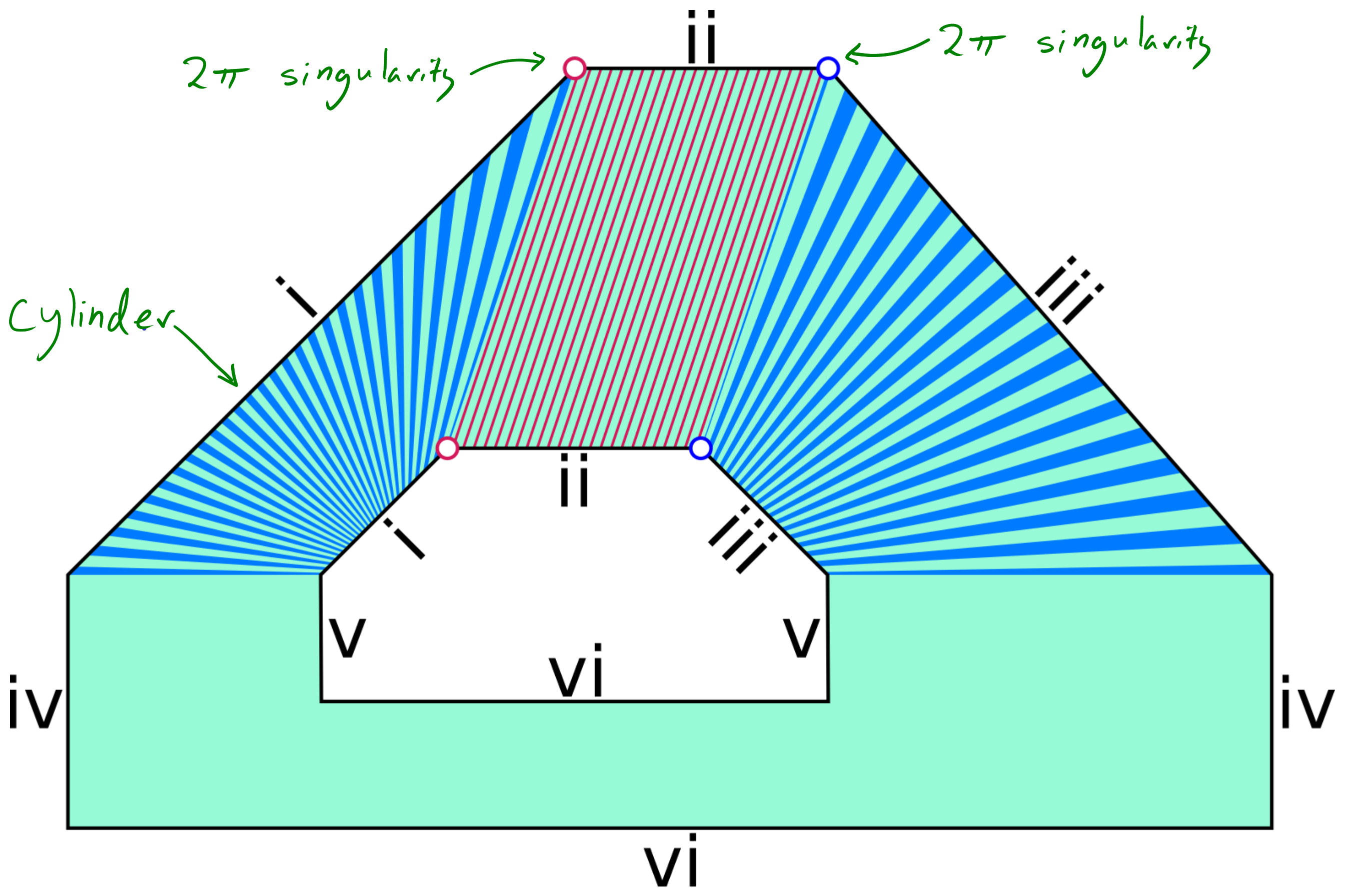
- ???



Cylinders

A cylinder is an annulus foliated by closed trails.



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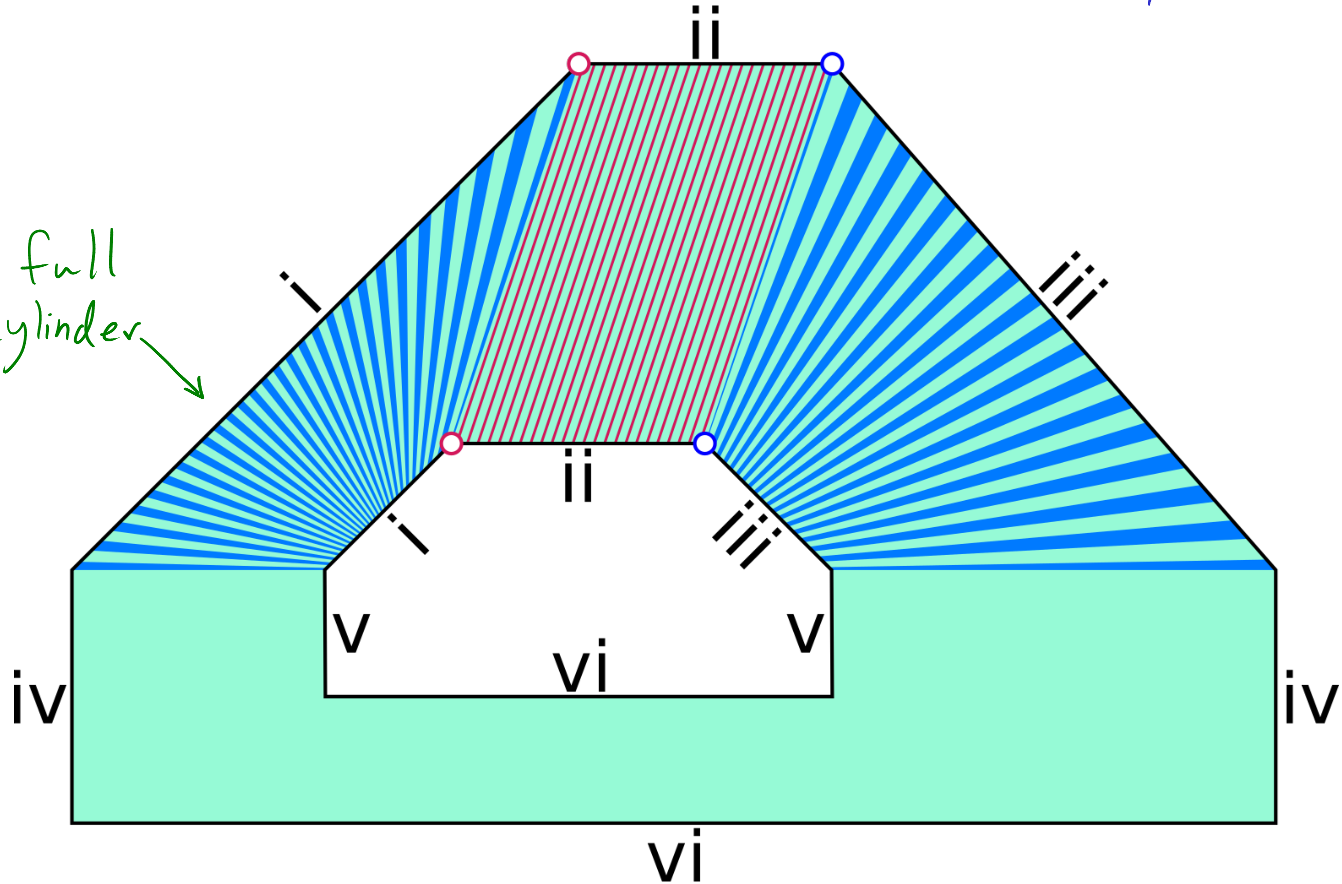
A cylinder is an annulus foliated by closed trails. A cylinder is full if it has leaves of all slopes.

full cylinder



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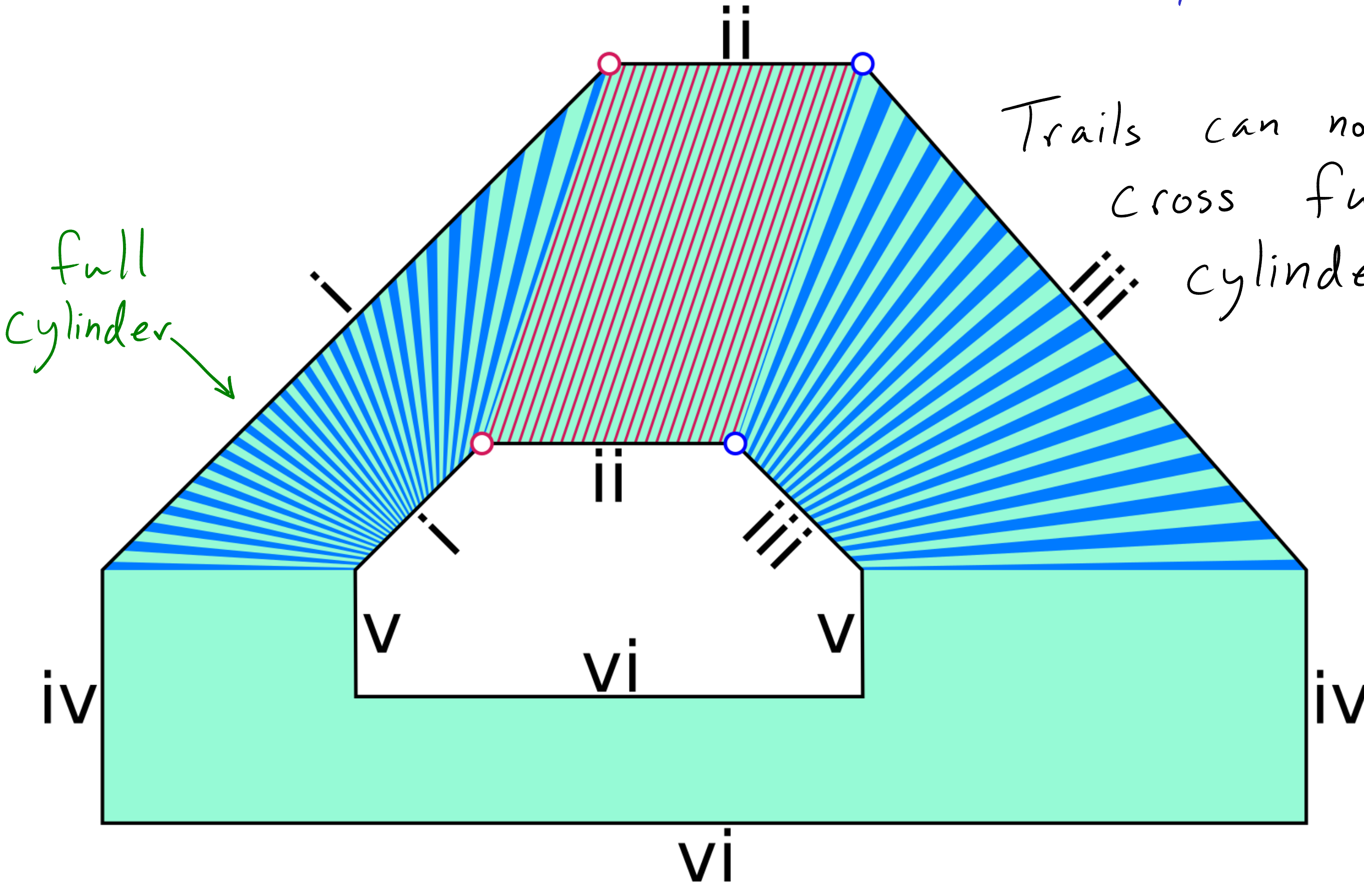
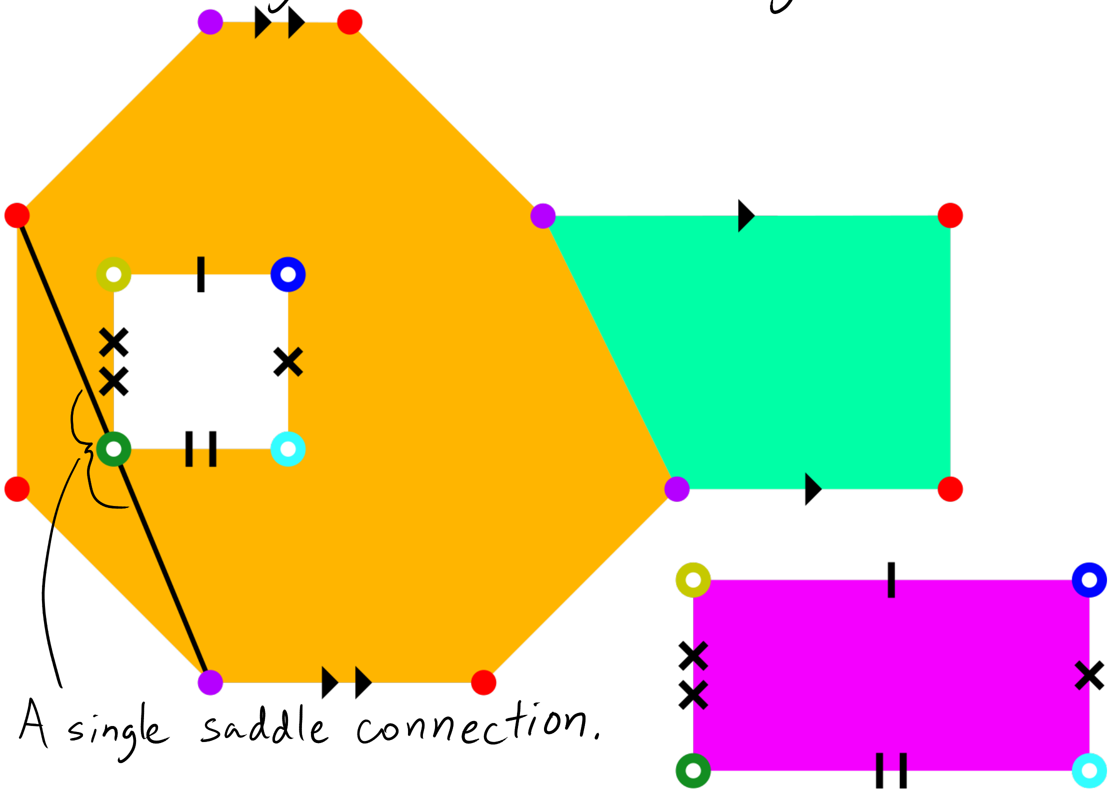


Trails can not cross full cylinders.

Definitions

- A saddle connection is a trail segment joining singularities with angle $\geq 3\pi$ that passes through no singularities with angle $\geq 3\pi$.



A single saddle connection.

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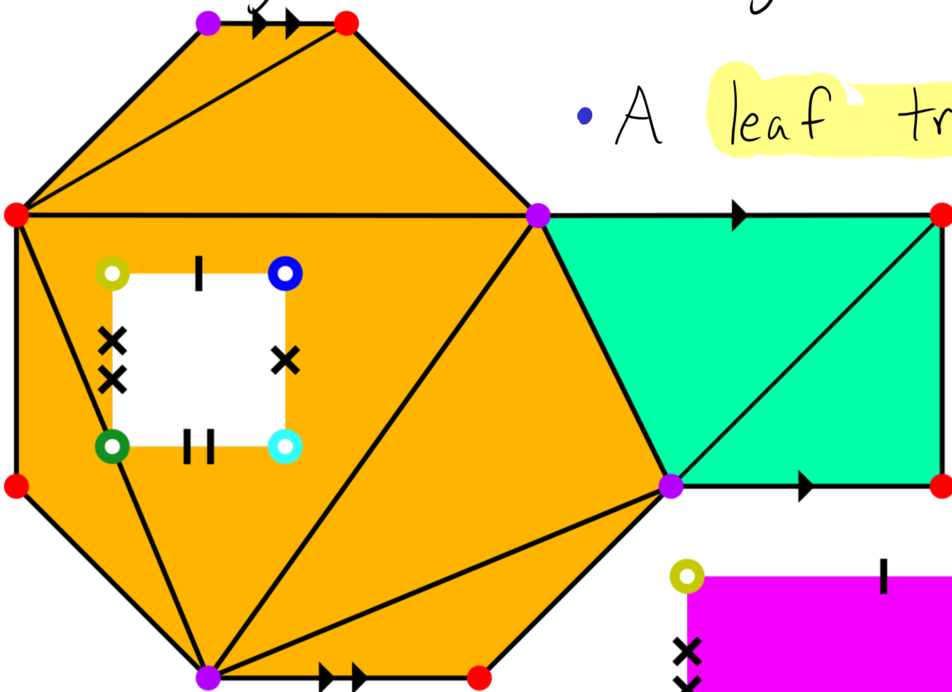
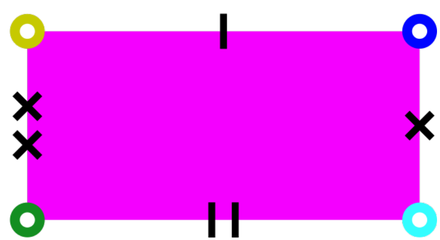


- A leaf triangulation is a triangulation whose edges are saddle connections.



Theorems (H - Valdez - Weiss)

Theorem 1. Let $\tilde{S}$ be the universal cover of a zebra surface. If $\tilde{S}$ has a leaf triangulation then $\tilde{S}$ is convex: Any two points can be joined by a trail.

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Theorem 2. If $\tilde{S}$ is convex, then every homotopy class of essential loops on S contains either a unique closed trail or there is a cylinder foliated by trails.

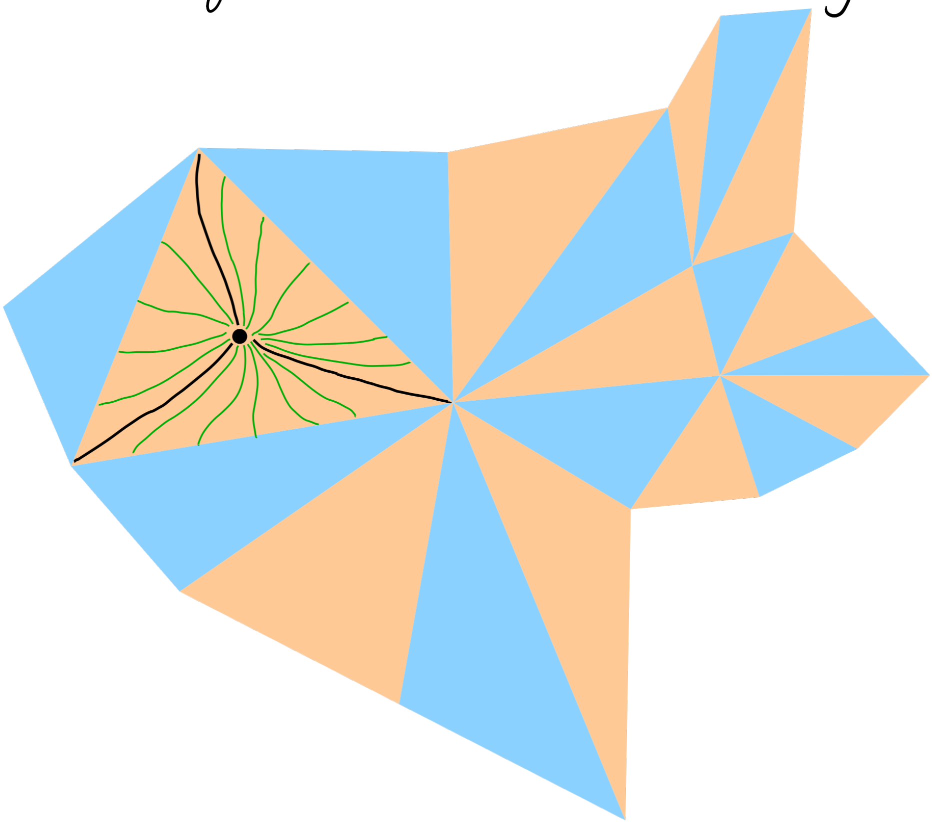
Theorem 3 (H-V-W)

Let S be a zebra structure on a closed surface. The following are equiv.

1. S has a leaf triangulation.
2. The universal cover $\tilde{S}$ is convex.
3. Every nontrivial homotopy class of closed curves is realized by a trail.
4. S contains no full cylinders.

Extremely abrieviated proof of Theorem 1

Choose a $p \in \tilde{S}$. Show trails emanating from p cover $\tilde{S}$ by induction on triangles. $\square$



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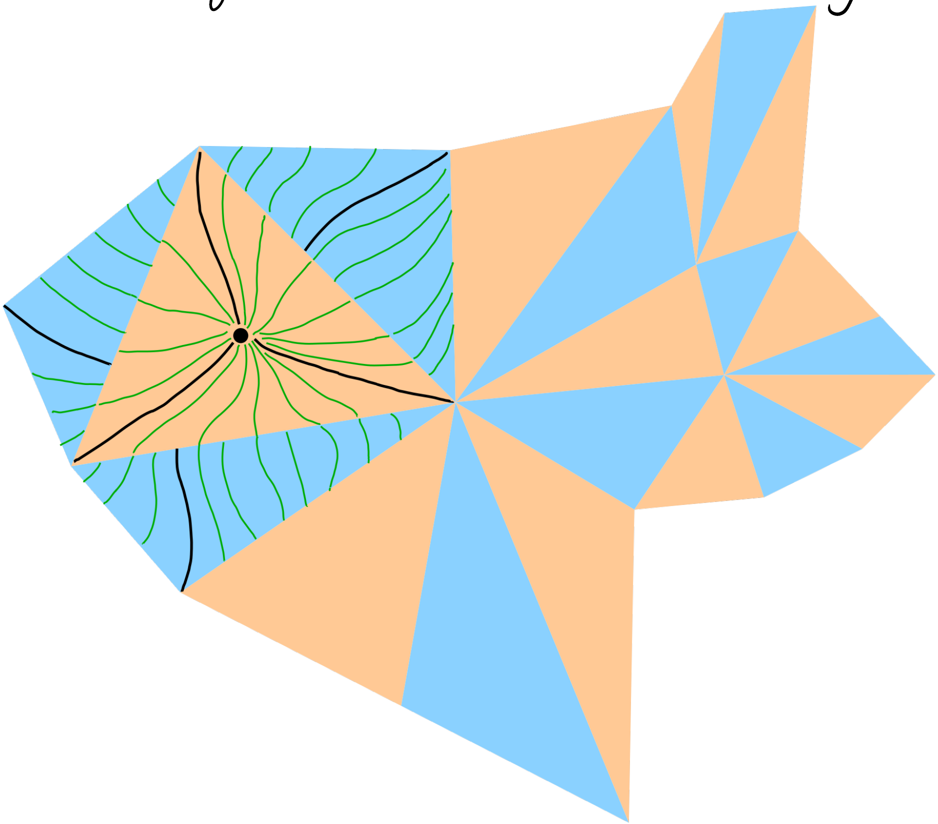

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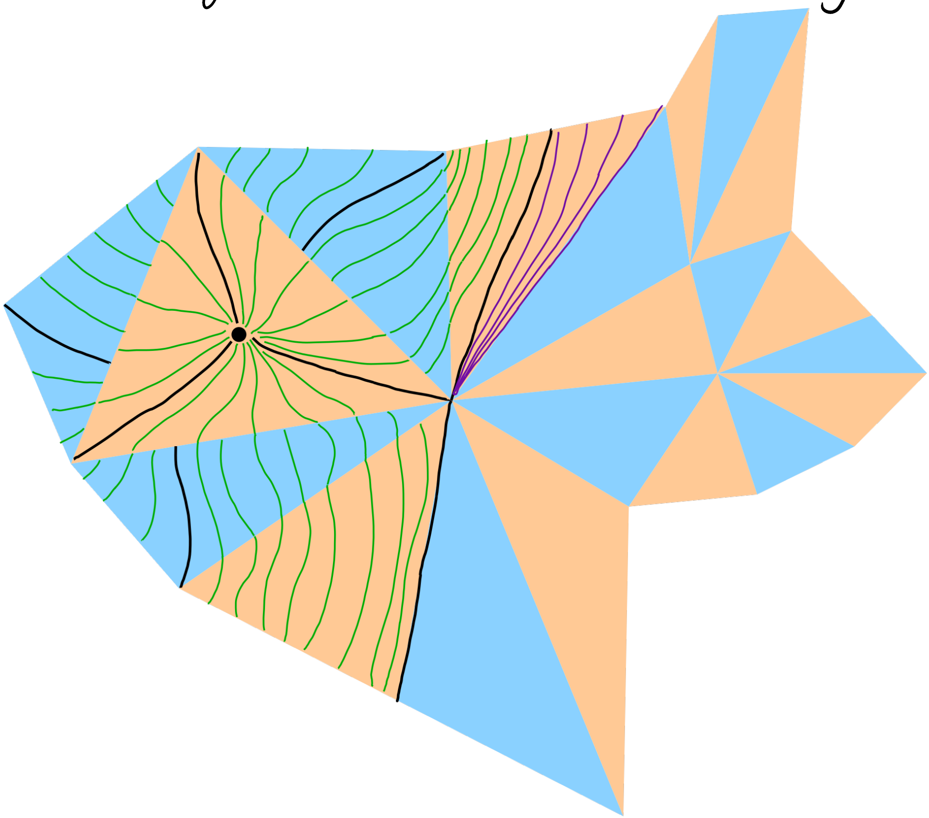

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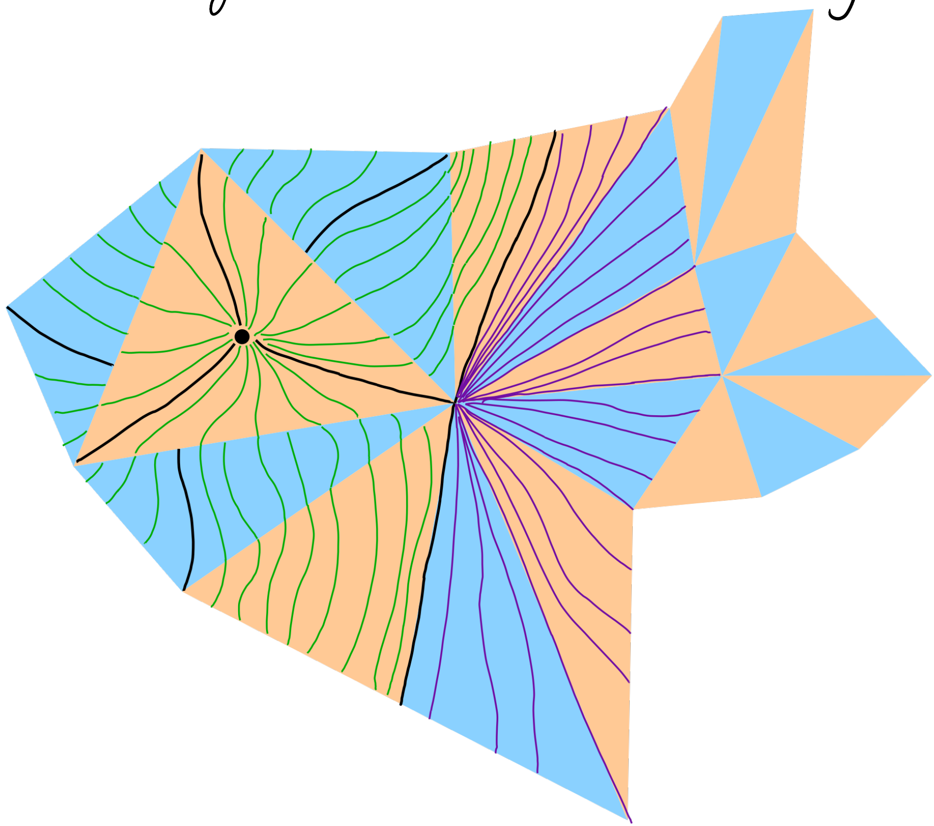

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- Is there a natural topology on the space of zebra structures up to isotopy?
- Can $SL(2,\mathbb{R})$ or $\mathrm{Homeo}_+(\widehat{\mathbb{R}})$ be used for renormalization of foliations?
- Are there zebra structures with interesting $\mathrm{Homeo}_+(\widehat{\mathbb{R}})$ stabilizers (up to homeomorphism of the surface)?