Geodesic representatives on surfaces without metrics

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joint work with Ferrán Valdez and Barak Weiss.

Talk Outline

1) Translation surfaces
2) Dilation surfaces
3) Metric geodesics / Trails
4) Zebra structures
5) Theorems on trails
6) Open questions.

Translation surfaces

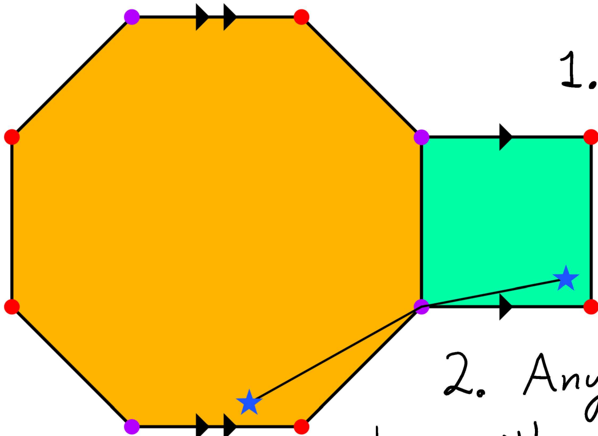
A translation surface is an topological surface with an atlas of charts to $\mathbb{R}^2$ such that transition functions are translations (and some singular points).

Objects associated to translation surfaces

- cone singularities (w/ angle $2k\pi$ for $k \in \mathbb{Z}_+$)
- a notion of direction (and slope)
- straight line flow in every direction
- an area measure (Lebesgue).
- measured foliations of all slopes.
- a metric (and so metric geodesics)
- an $SL(2,\mathbb{R})$ action on the space of surfaces.

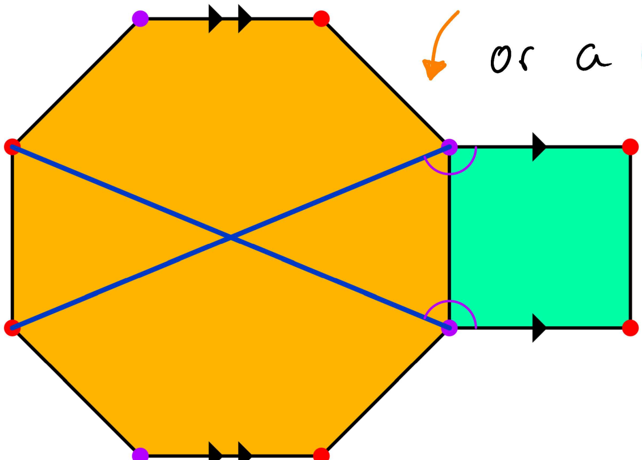
Metric geodesics in translation surfaces

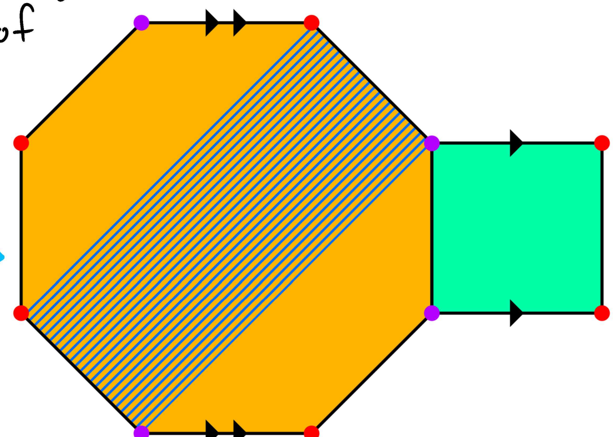
In a closed translation surface...



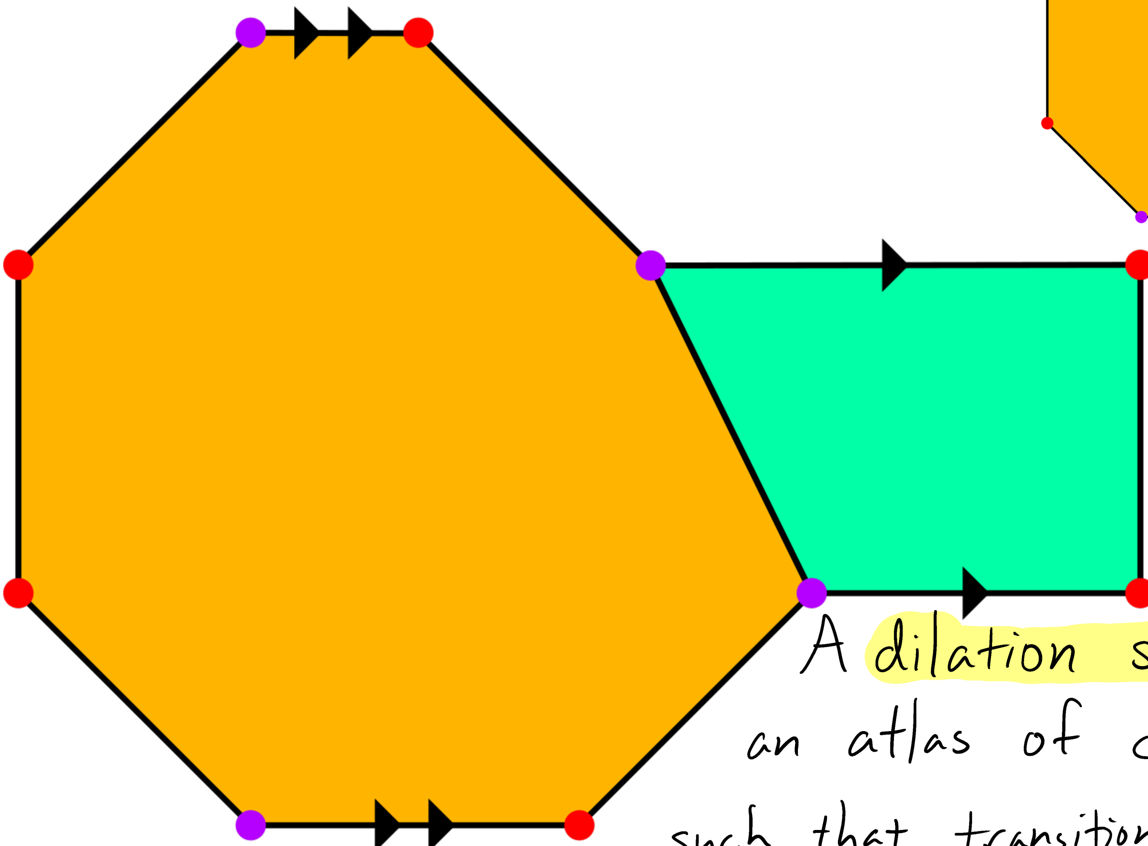
1. Any homotopy class of arcs rel endpoints has a metric geodesic representative.

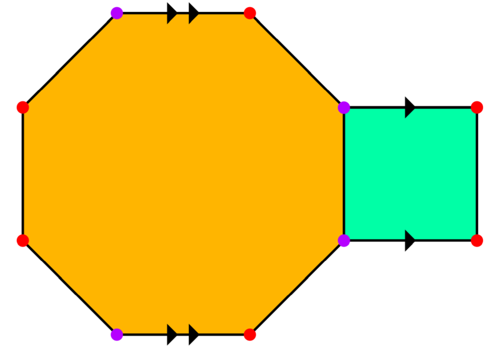
2. Any non-trivial free homotopy class of loops has either a unique bent geodesic representative or a cylinder of closed leaves.





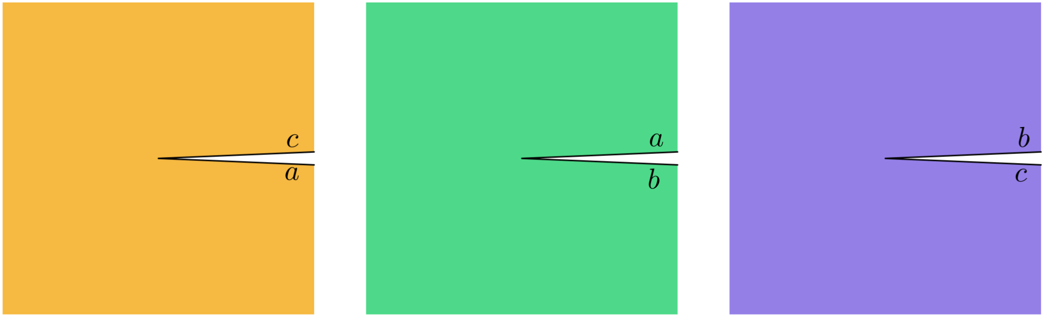
Dilation surfaces





A dilation surface has an atlas of charts to $\mathbb{C}$ such that transition functions are in $\{z \mapsto az+b : a \in \mathbb{R}_+ \text{ and } b \in \mathbb{C}\}$ (and singularities).

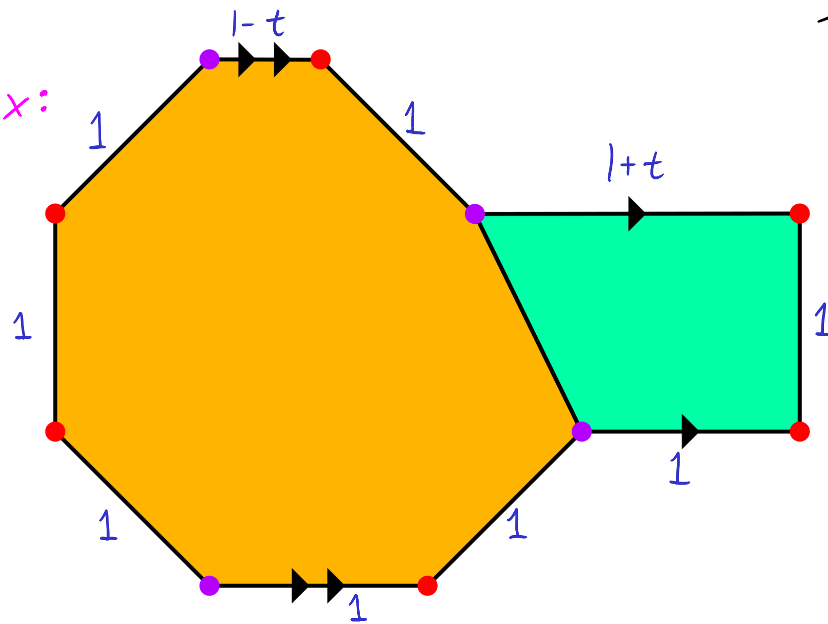
Dilation surface singularities



Model:

Slit planes $\mathbb{C} \setminus \mathbb{R}_+$ glued cyclically by dilations along boundary rays.

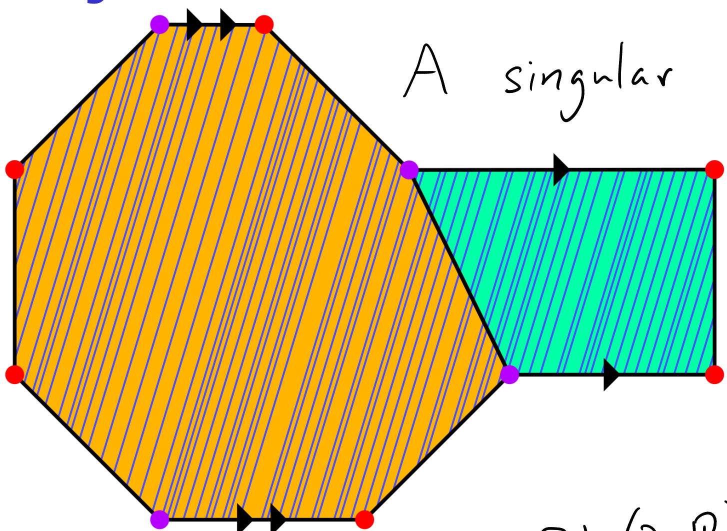
Ex:



The red singularity has angle 4π and (counterclockwise) dilation $(1+t)(1-t) = 1-t^2$.

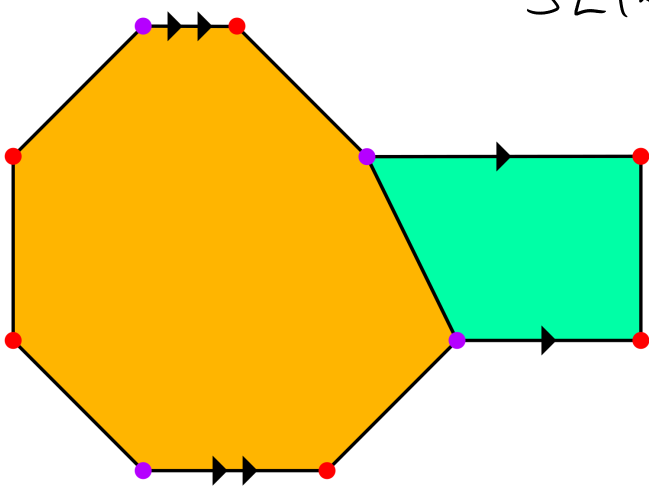
The purple singularity has angle 4π and dilation $\frac{1}{1-t^2}$.

Objects associated to dilation surfaces

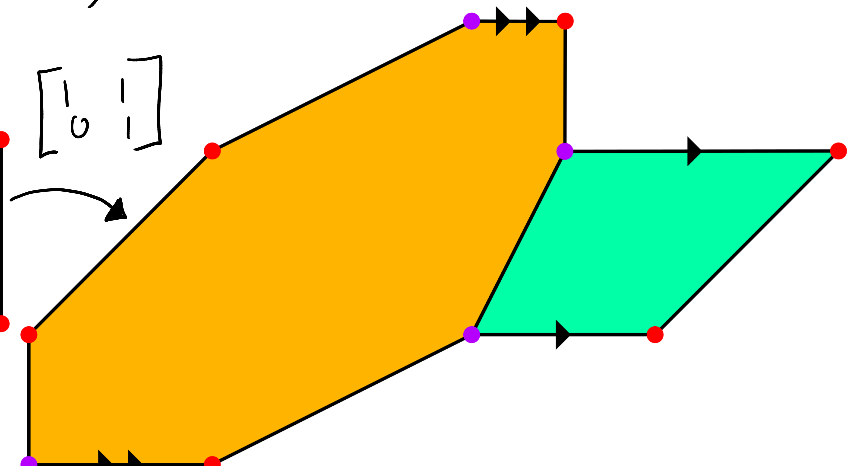


A singular foliation of every slope.

$SL(2,\mathbb{R})$-action on dilation surfaces.

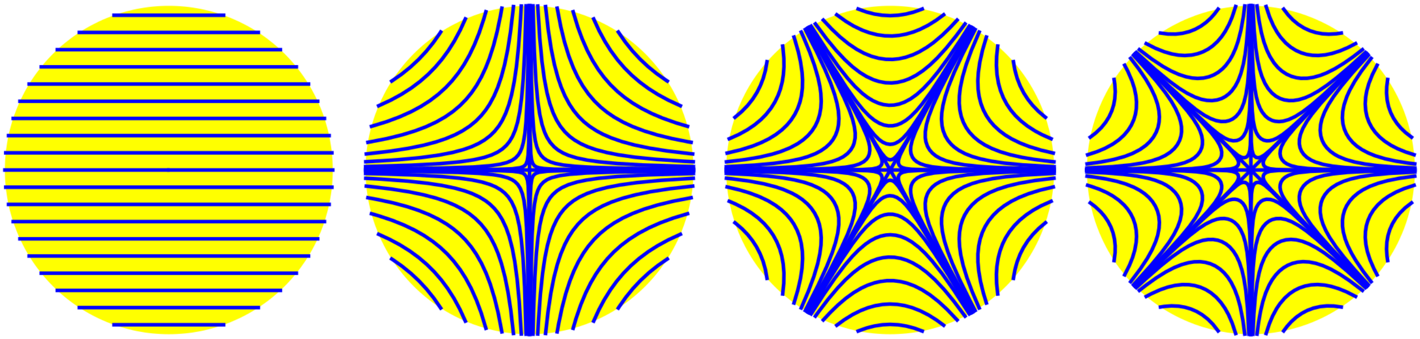


$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

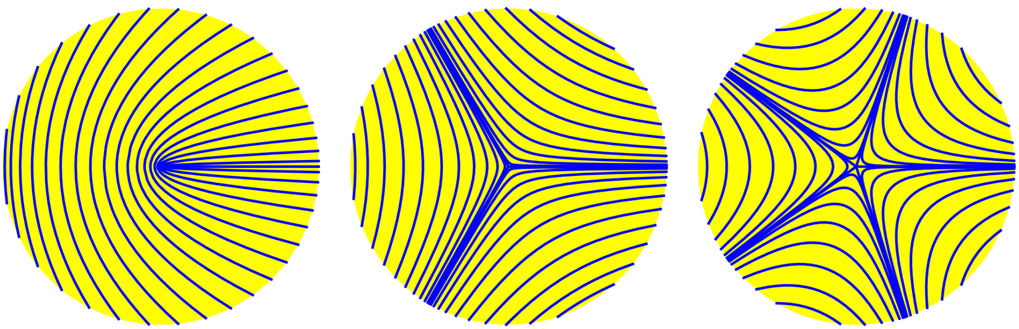


Singular foliations

Local models for an orientable singular foliation:



If foliations are not required to be orientable we also allow:



Translation Surfaces versus Dilation Surfaces

Translation surfaces have:	Dilation surfaces have:

Translation surfaces have:

1. A metric.
2. Lebesgue measure.
3. $GL(2,\mathbb{R})$ action on the space of surfaces.
4. A notion of direction or slope $m \in \hat{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.
5. Measured foliations of all slopes
6. Straight line flow in every direction
7. Metric geodesics

Dilation surfaces have:

1. No natural metric.
2. No natural Borel measure.
3. $SL(2,\mathbb{R})$ acts.
4. Yes!
5. Singular foliations instead.
6. No flow; unparametrized leaves.
7. "Trails" ← Main topic of talk.

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Zebra surfaces have:

1. } Same
2. } issues.
3. Better: $\text{Homeo}(\hat{\mathbb{R}})$ acts.
4. Yes, same.
5. Same.
6. Same
7. "Trails" here too.

Topics studied related to Dilation Surfaces:

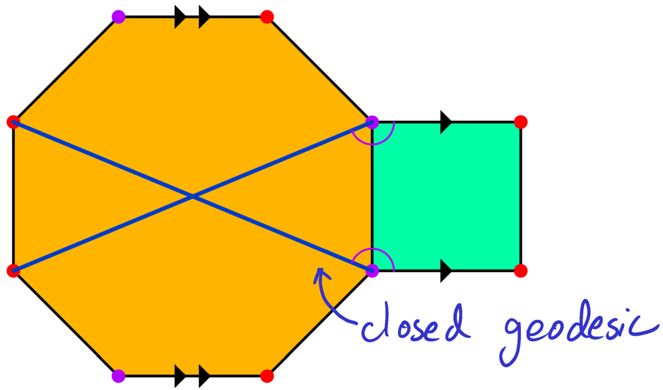
- Algebraic structure of moduli spaces (Veech, Apisa - Bainbridge - Wang)
- Affine symmetry groups (Duryev - Fougeron - Ghazouani)
- Affine realization of mapping classes (Wang)
- Dynamics of directional foliations (Liousse, Bowman - Sanderson, Boulanger - Fougeron - Ghazouani)
- Existence of closed leaves (Boulanger - Ghazouani - Tahar)

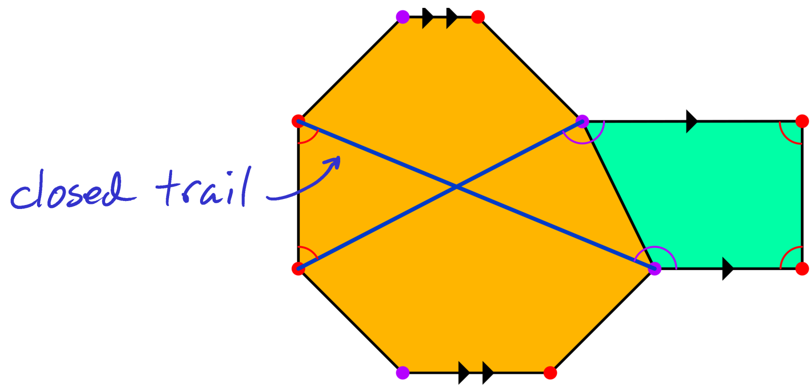
Related ideas:

- Affine interval exchange maps (Camelier - Gutierrez, Cobo, Cobo - Gutiérrez-Romo - Maass, Marmi - Moussa - Yoccoz, ...)
- Twisted measured laminations (McMullen, for studying fibered 3-manifolds)
- Infinite translation surfaces (Hooper - Hubert - Weiss)

Trails in dilation surfaces

A trail in a dilation surface is a maximal bi-infinite path that follows leaves (maximal line segments), transitioning between leaves only at singularities in such a way so that the two angles made at the singular transitions measure at least π.





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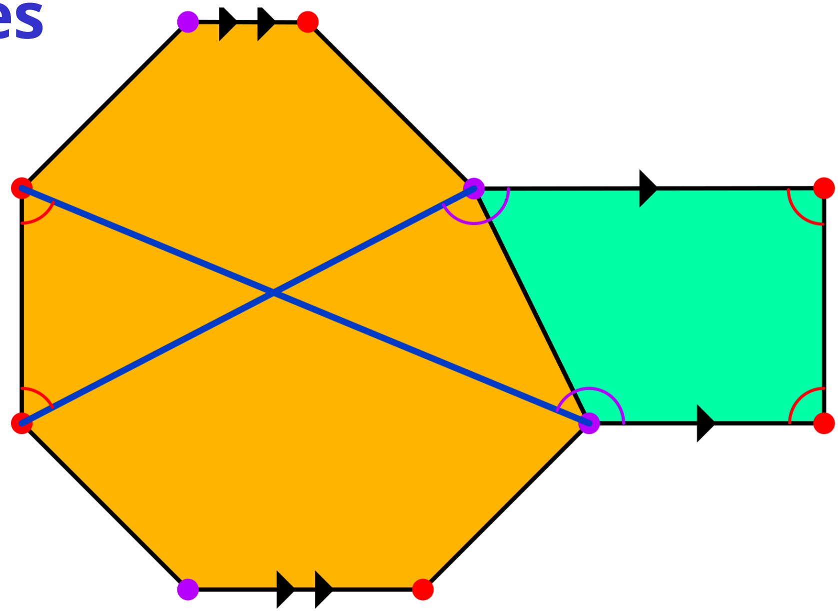
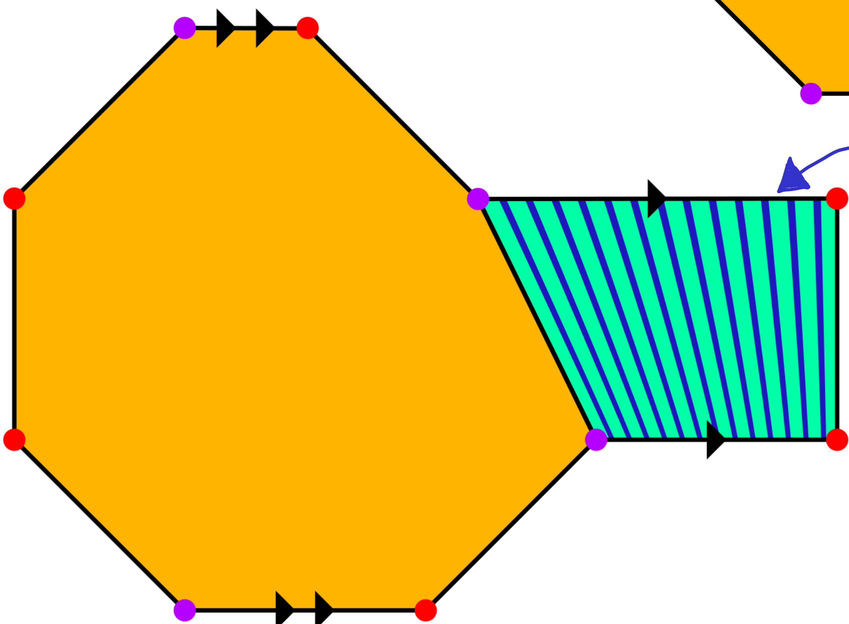
Main goal:

State a theorem guaranteeing when a homotopy class can be guaranteed to have a trail representative.

Trail Representatives

A homotopy class of closed curves represented by a unique trail.

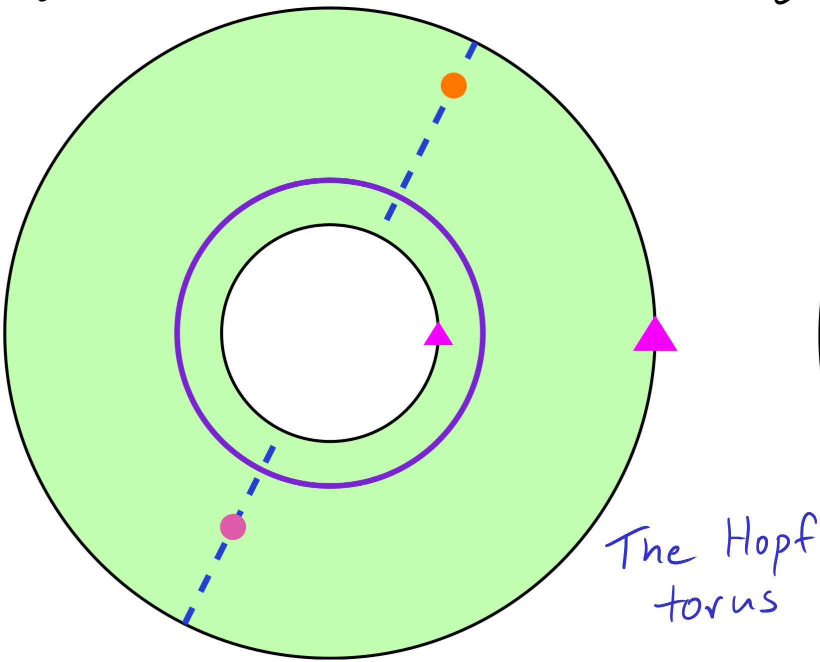


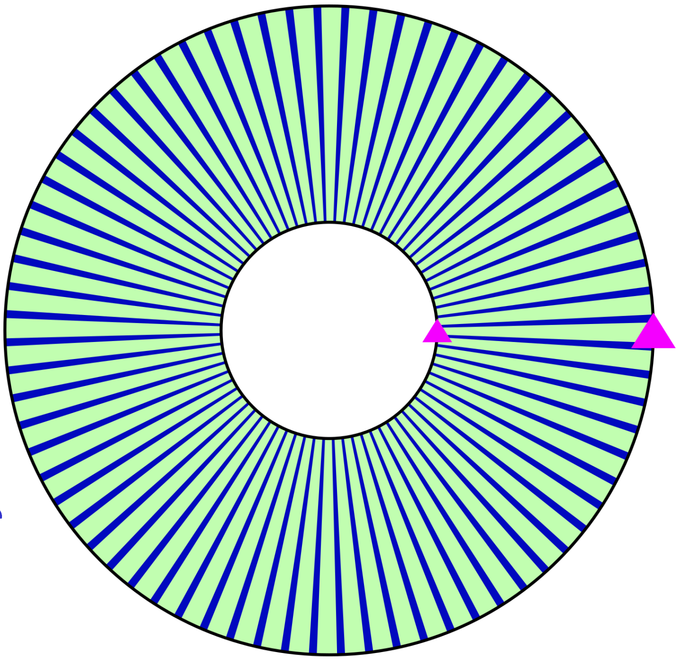


A dilation cylinder representing a homotopy class of closed curves.

The Hopf torus as a counterexample

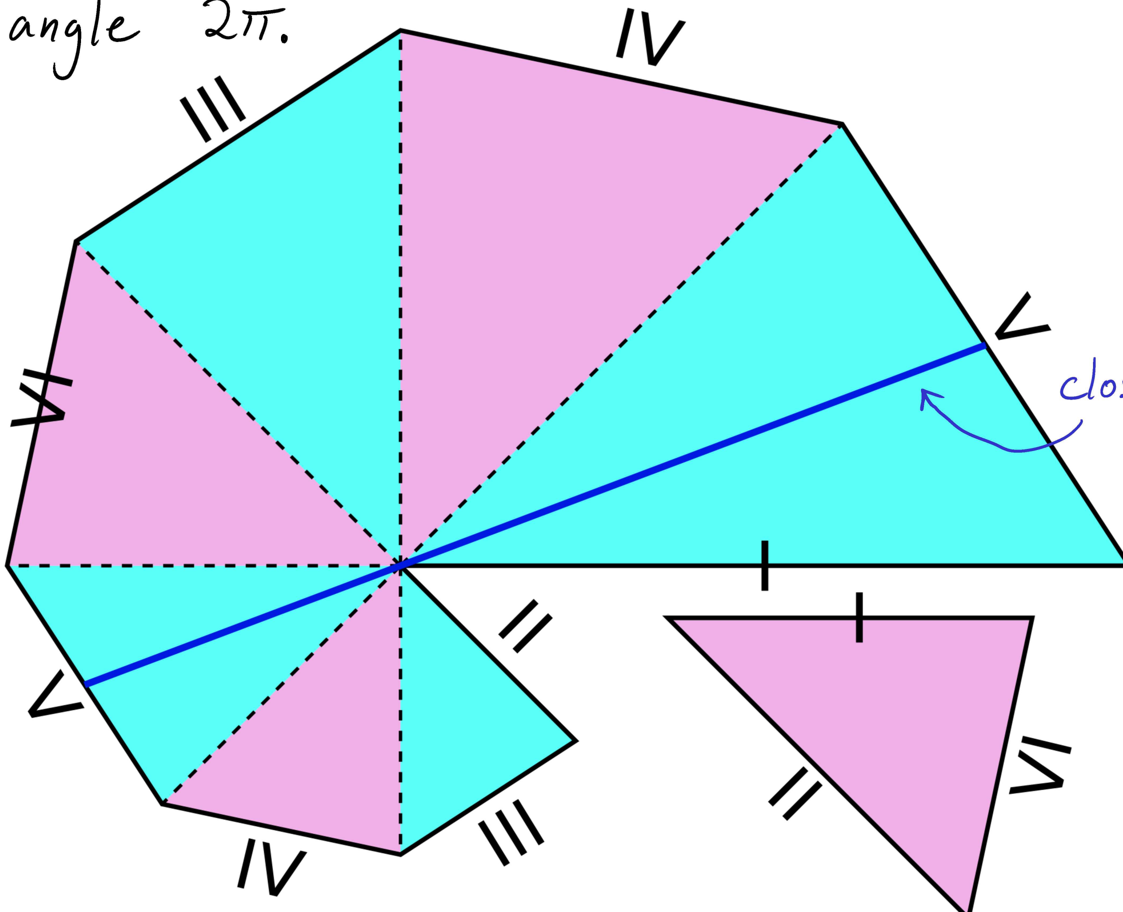
In the Hopf torus, the homotopy class of the purple loop can not be realized as a closed trail, and no trail segment joins the pink and orange points.





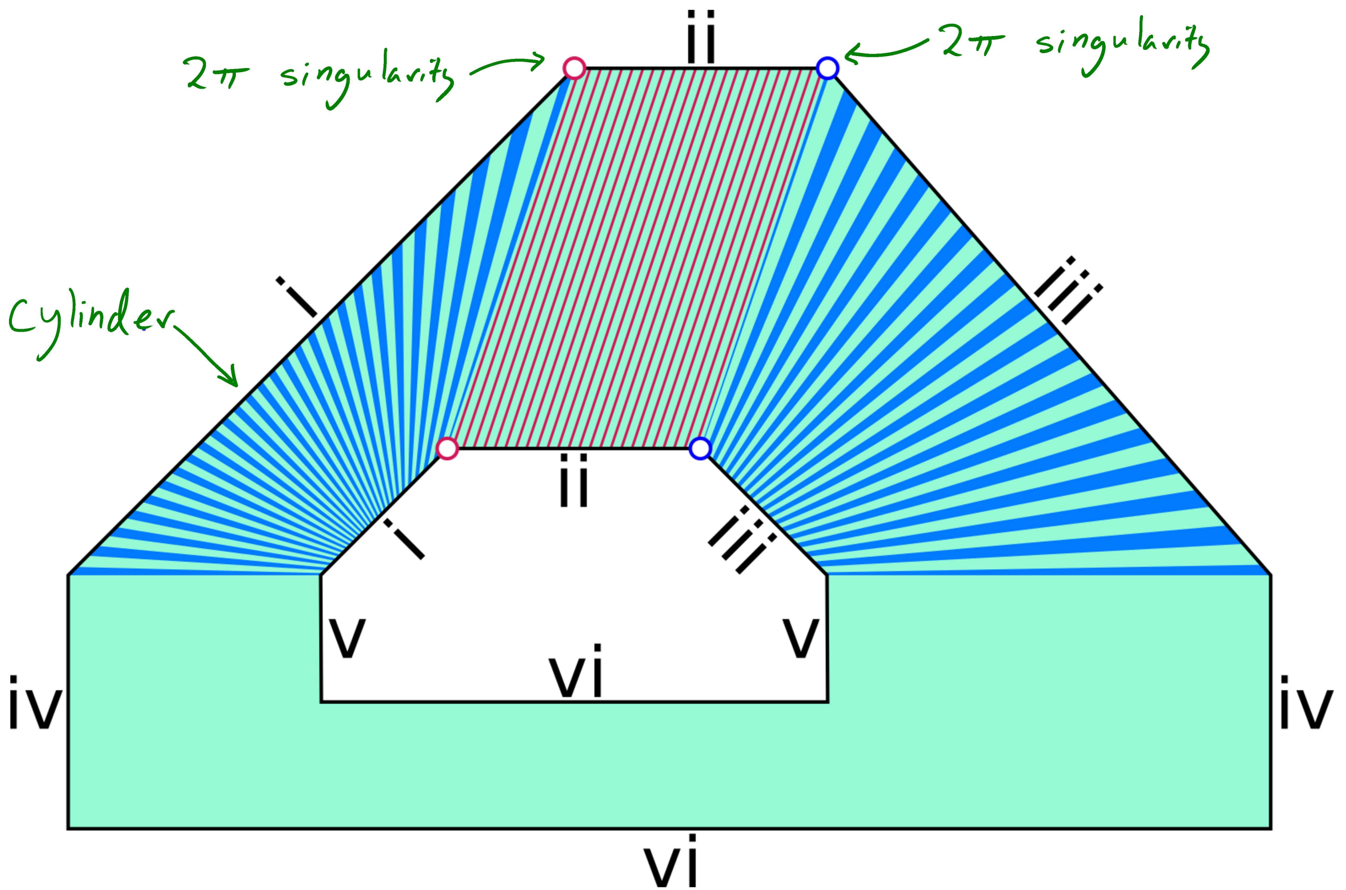
More trouble: 2π dilation singularities... but we like trouble!

Trails pass straight through dilation singularities with angle 2π.



Cylinders

A cylinder is an annulus foliated by closed trails.



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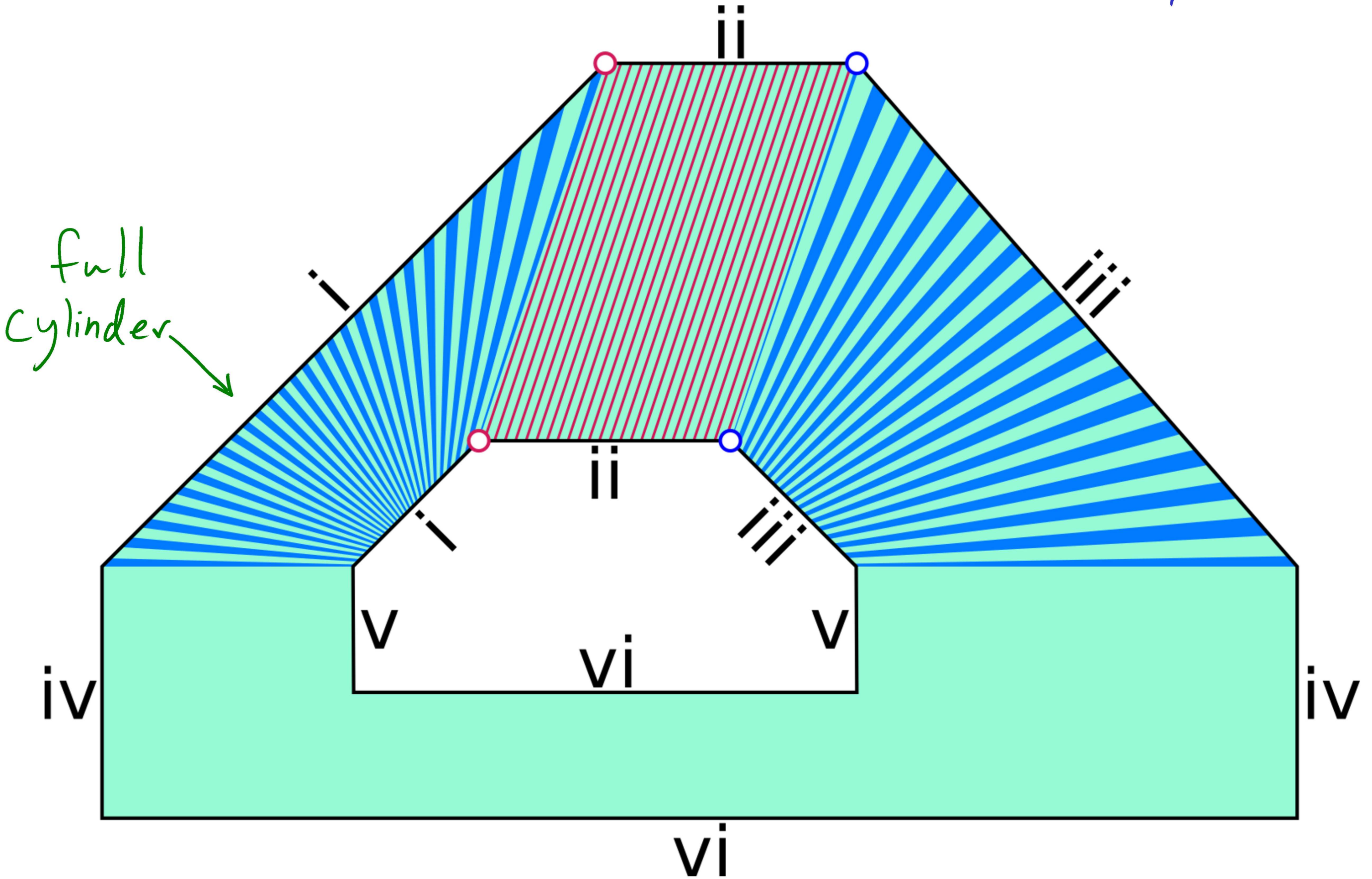
A cylinder is an annulus foliated by closed trails. A cylinder is full if it has leaves of all slopes.



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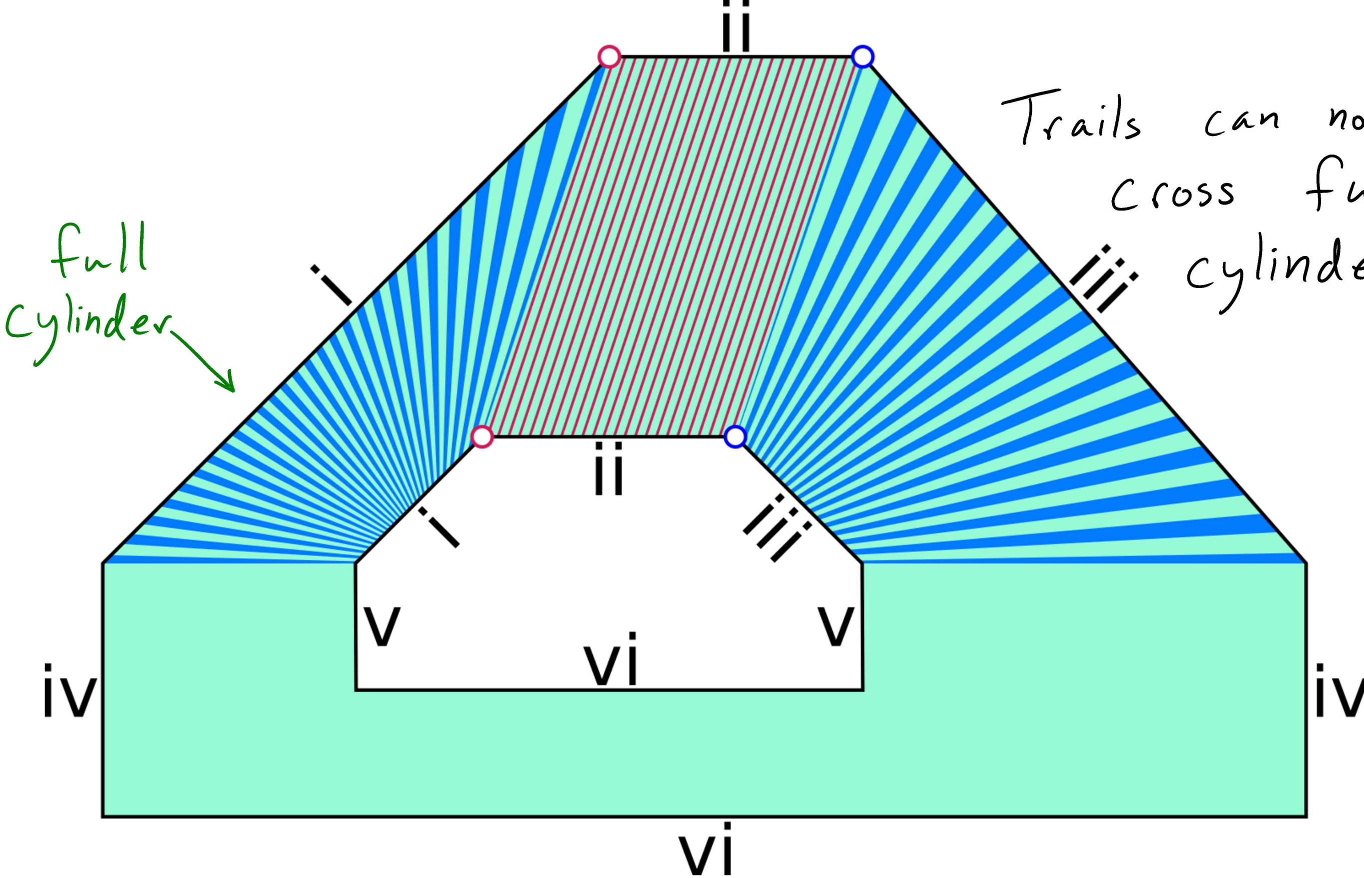
Trails can not cross full cylinders.



Philosophical issues:

Dilation singularities with angle 2π should be treated like regular points.

Gauss-Bonnet allows arbitrarily many...

But they do distort the geometry... So...

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But they do distort the geometry... So...

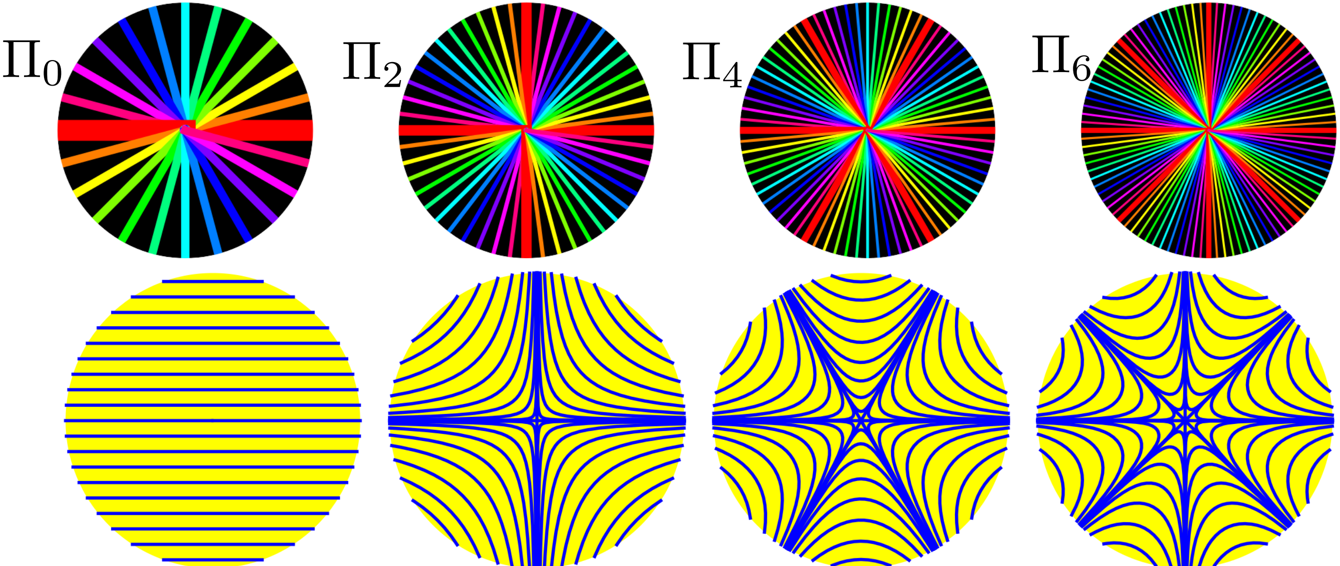
we declare them regular points and throw out the geometry (modeling on $\mathbb{C}$).

Stellated foliation/zebra structures

Let S be an oriented topological surface and let $\{\mathcal{F}_m : \ m \in \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}\}$ be a collection of singular foliations indexed by slope.

We say that $\{\mathcal{F}_m\}$ is a zebra structure if:

> For each point p in S, there is an open neighborhood N containing p and a homeomorphism from N to a model space Π_k such that for all $m \in \hat{\mathbb{R}}$, the homeomorphism induces a bijection between the prongs of $\mathcal{F}_m$ at p and the rays of slope m in Π_k.



Examples of zebra structures

- Translation and dilation structures give rise to zebra structures.

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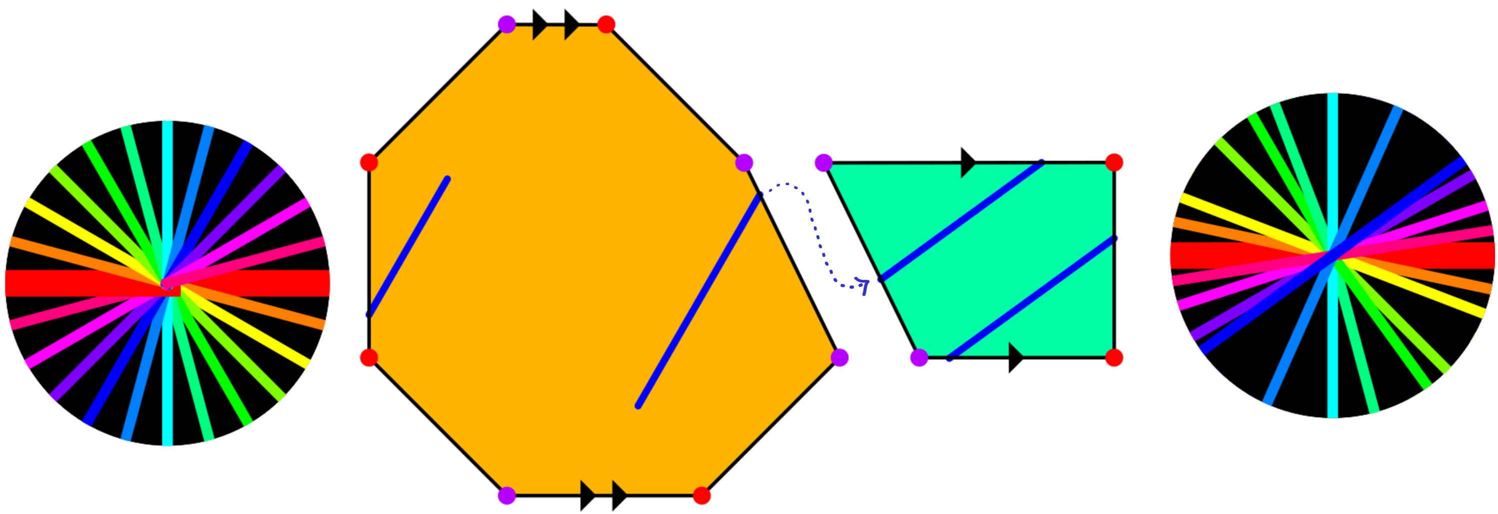

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- Surfaces formed by gluing parallel edges of a union of polygons together by homeomorphism.
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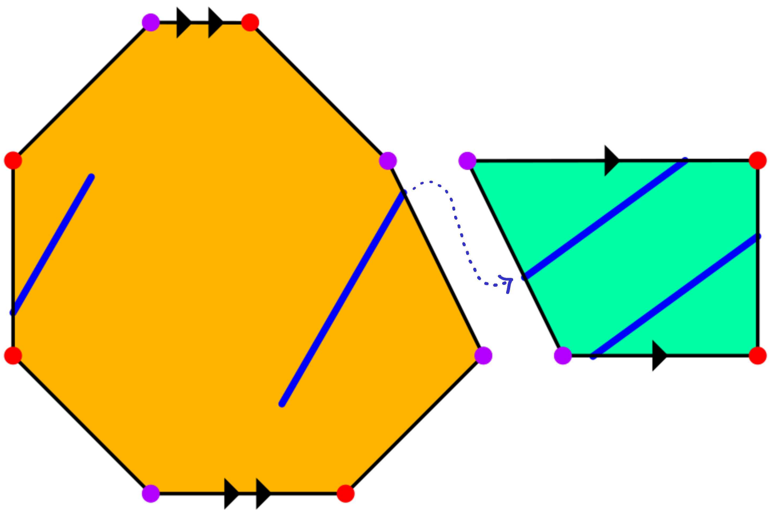
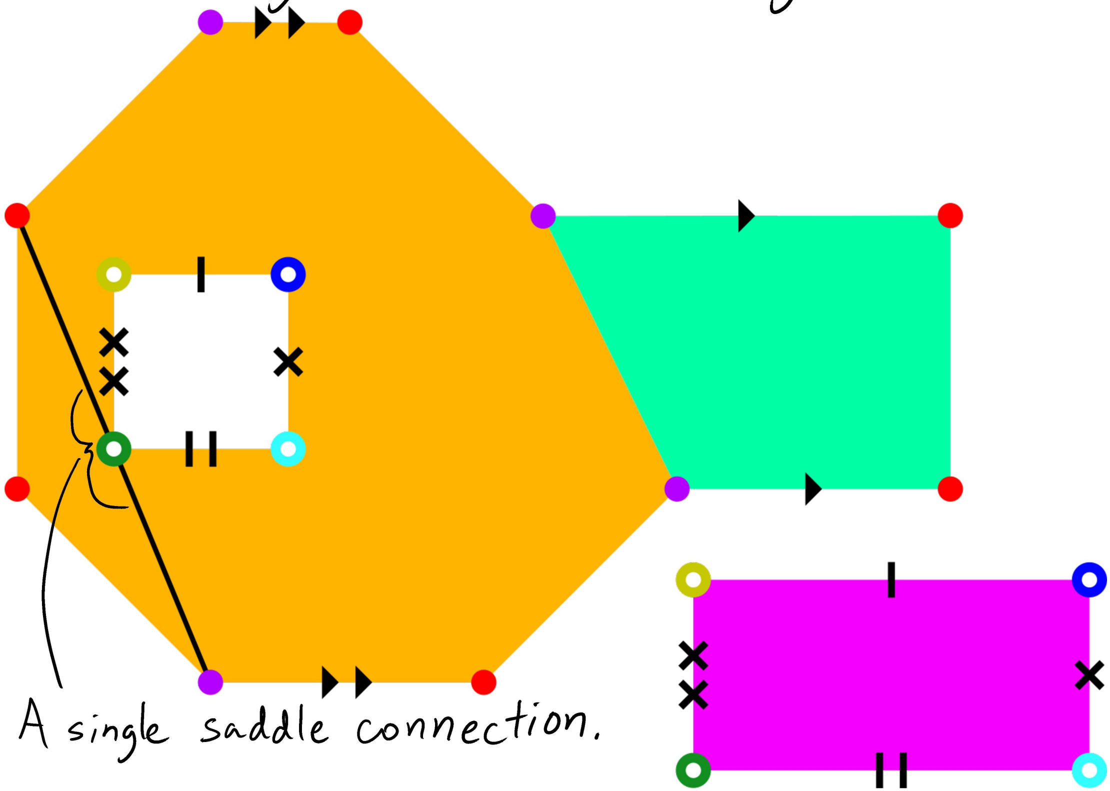
- ???


Definitions

- A saddle connection is a trail segment joining singularities with angle $\geq 3\pi$ that passes through no singularities with angle $\geq 3\pi$.



A single saddle connection.

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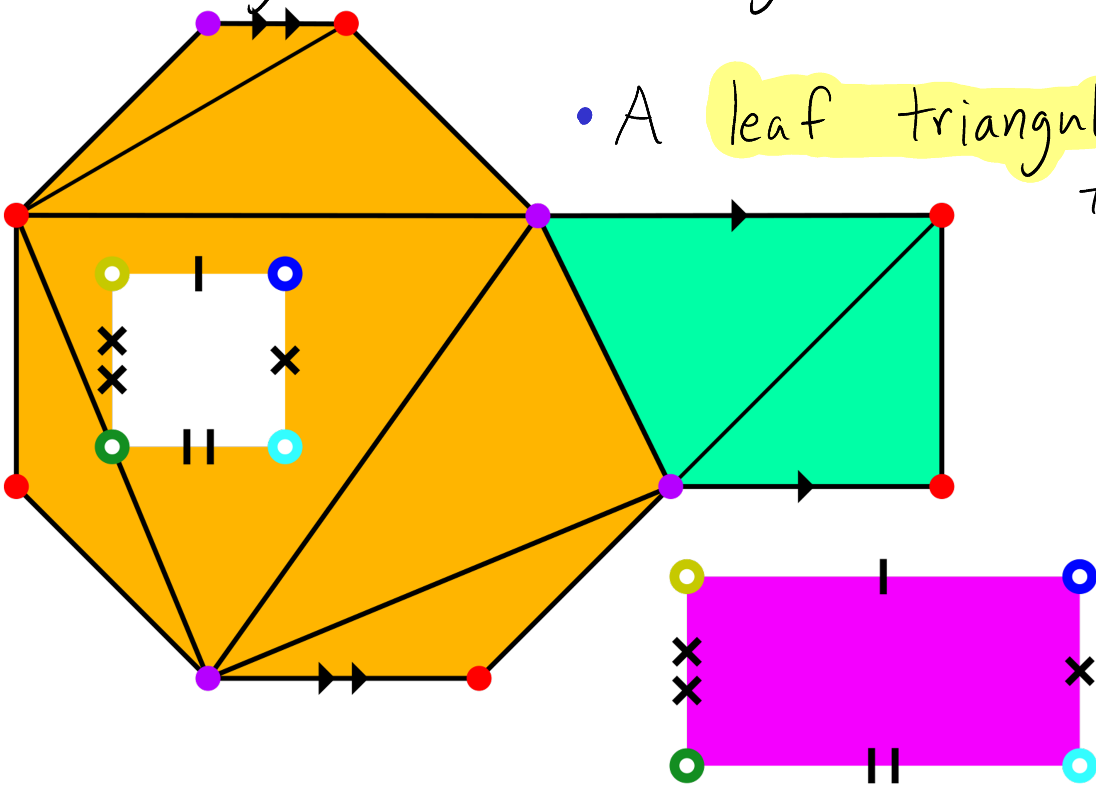

- A leaf triangulation is a triangulation whose edges are saddle connections.

Theorems (H-Valdez-Weiss)

Theorem 1. Let $\tilde{S}$ be the universal cover of a zebra surface. If $\tilde{S}$ has a leaf triangulation then $\tilde{S}$ is convex: Any two points can be joined by a trail.

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Theorem 1. Let $\tilde{S}$ be the universal cover of a zebra surface. If $\tilde{S}$ has a leaf triangulation then $\tilde{S}$ is convex: Any two points can be joined by a trail.

Theorem 2. If $\tilde{S}$ is convex, then every homotopy class of essential loops on S contains either a unique closed trail or there is a cylinder foliated by trails.

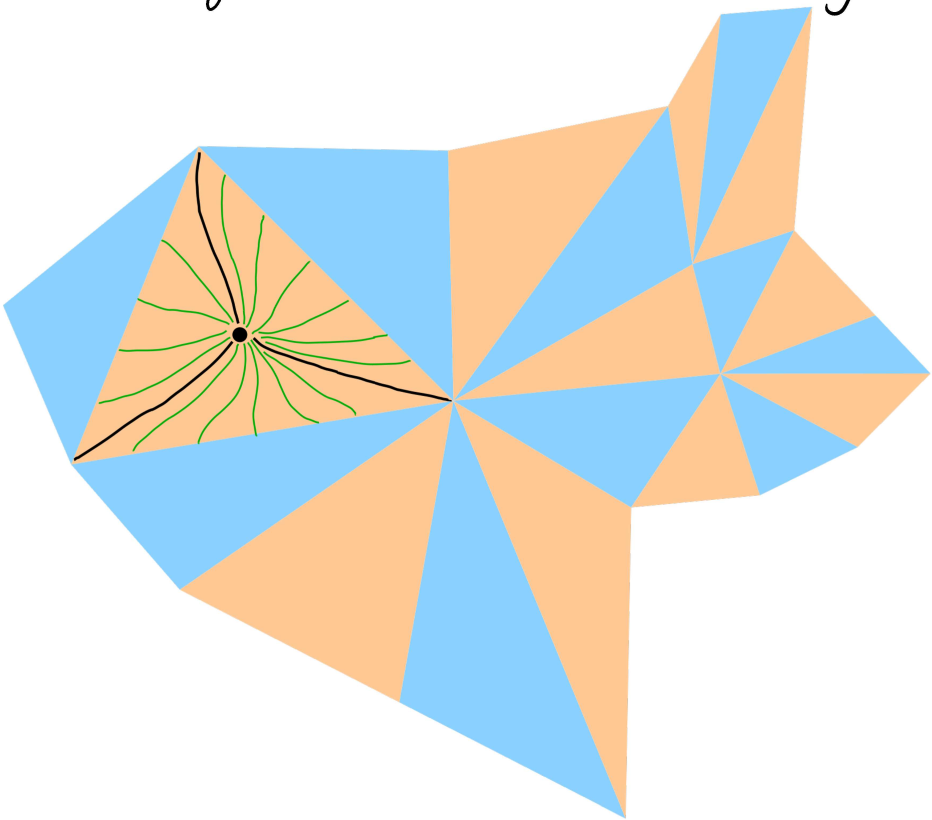
Theorem 3 (H-V-W)

Let S be a zebra structure on a closed surface. The following are equiv.

1. S has a leaf triangulation.
2. The universal cover $\tilde{S}$ is convex.
3. Every nontrivial homotopy class of closed curves is realized by a trail.
4. S contains no full cylinders.

Extremely abrieviated proof of Theorem 1

Choose a $p \in \tilde{S}$. Show trails emanating from p cover $\tilde{S}$ by induction on triangles. $\square$



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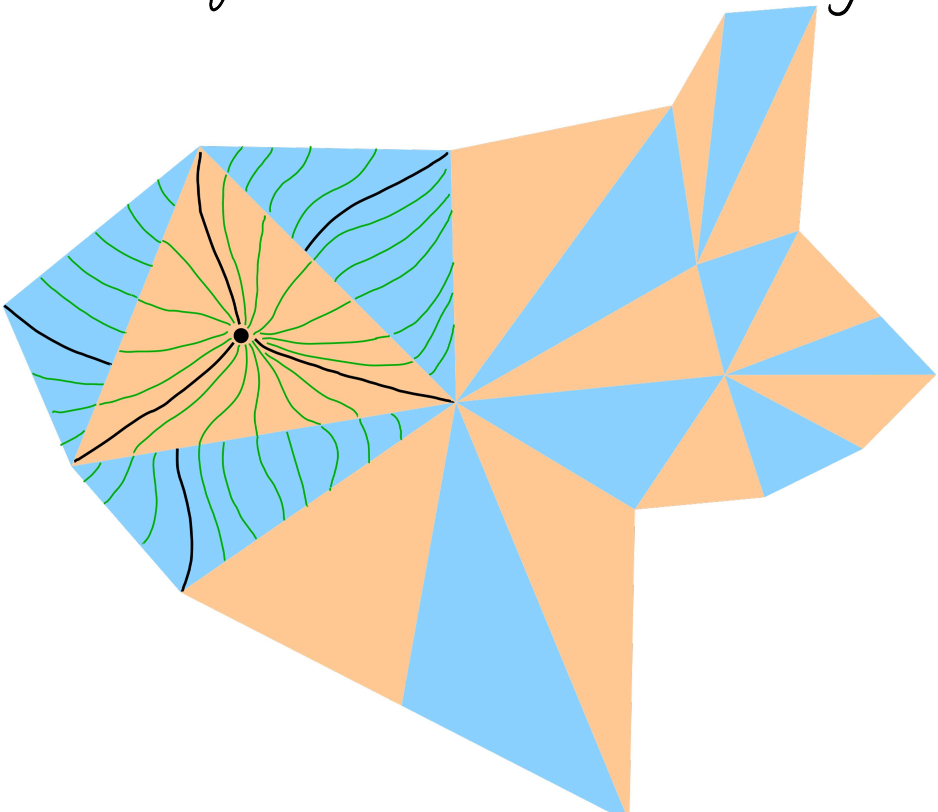

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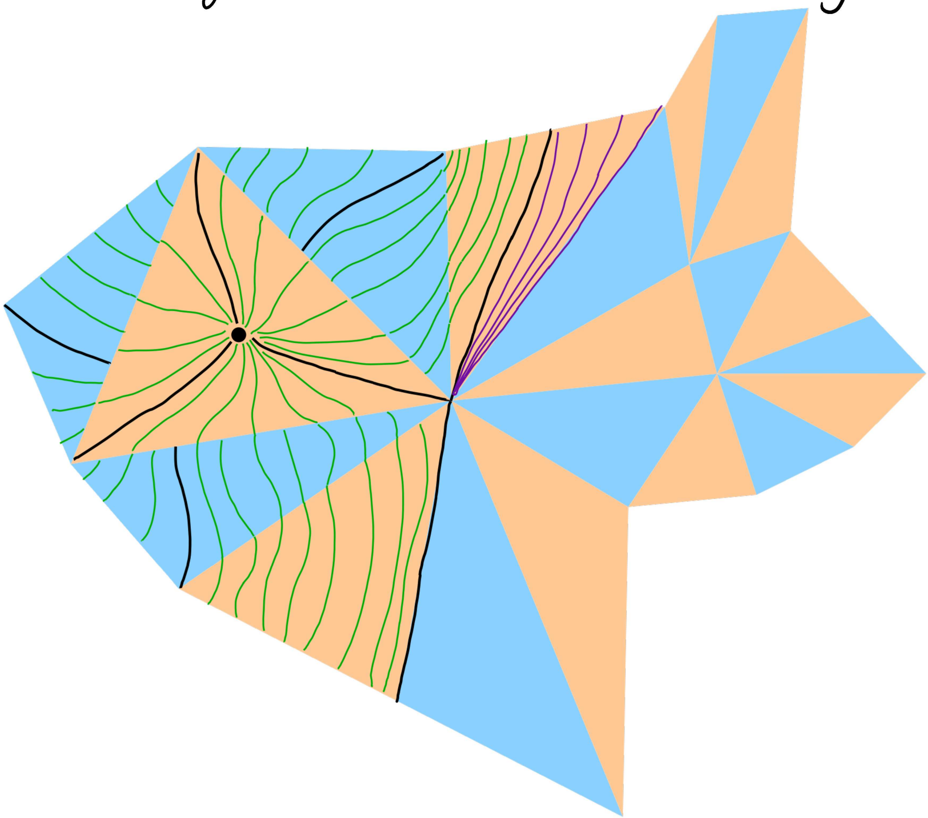

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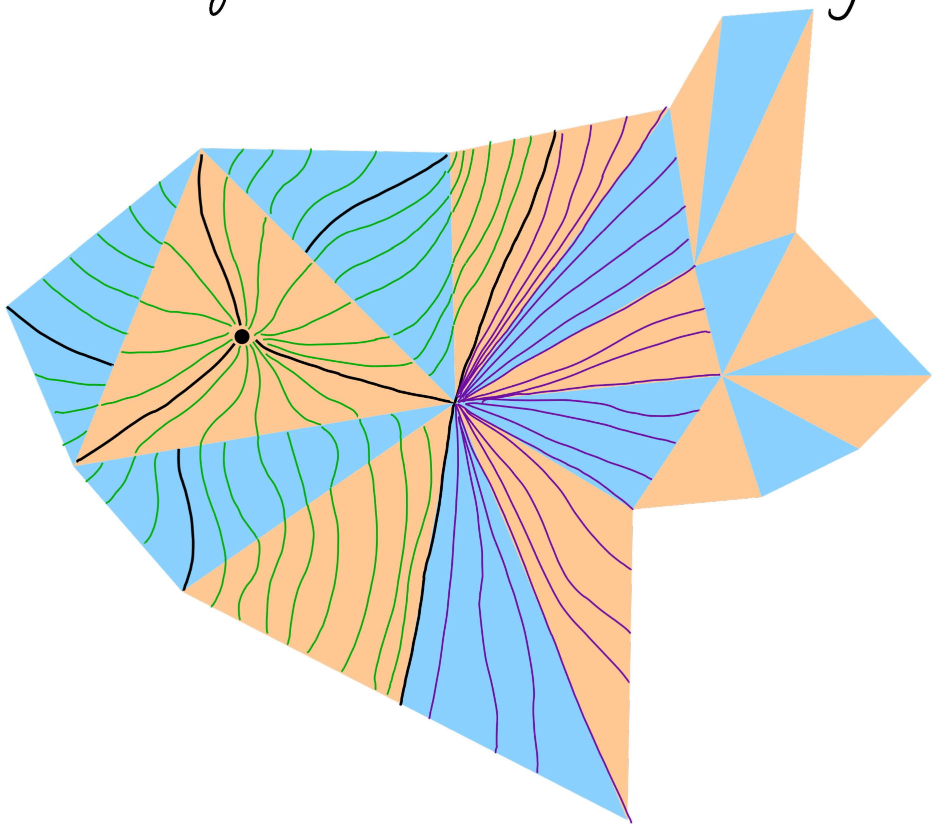

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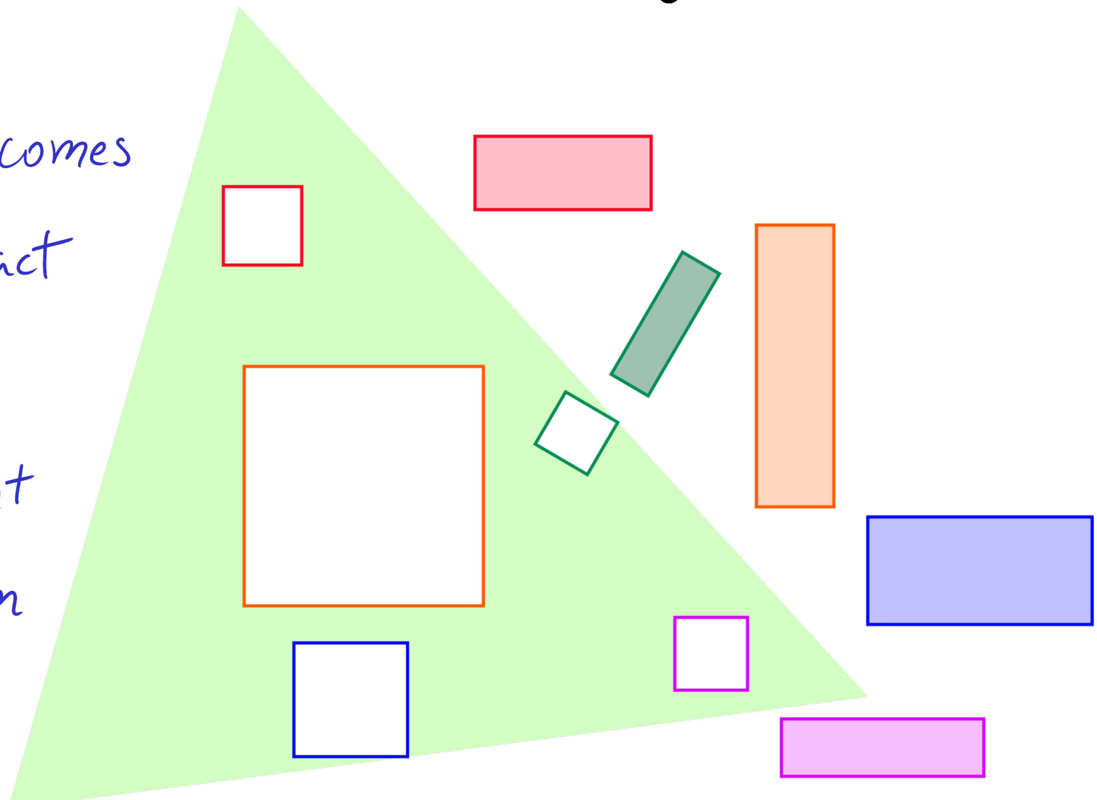
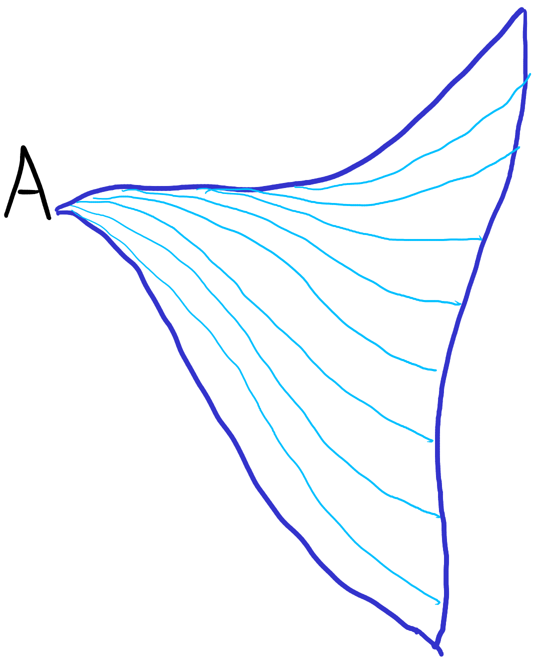
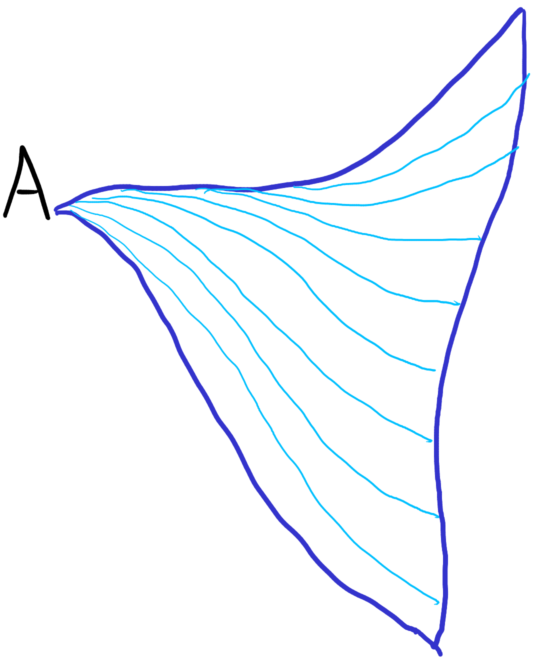
One hurdle comes from the fact that our triangles might be far from Euclidean.

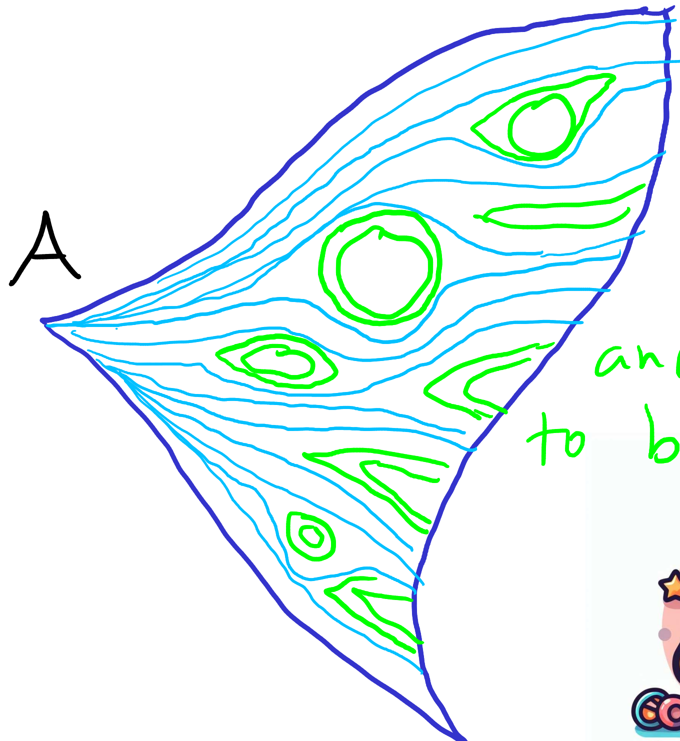


Ingredient: Given a triangle ABC in a zebra surface, the leaves emanating from A foliate $\triangle \underline{A}BC$.

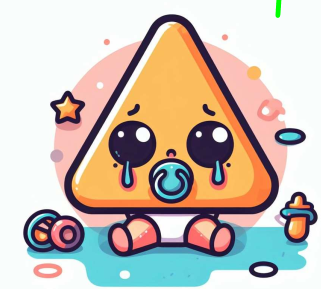


Ingredient: Given a triangle ABC in a zebra surface, the leaves emanating from A foliate $\triangle ABC$.

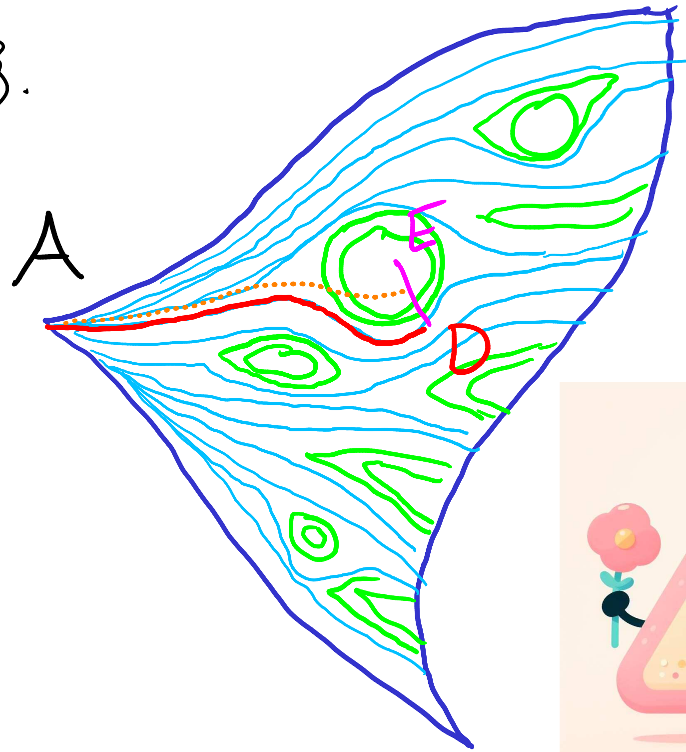


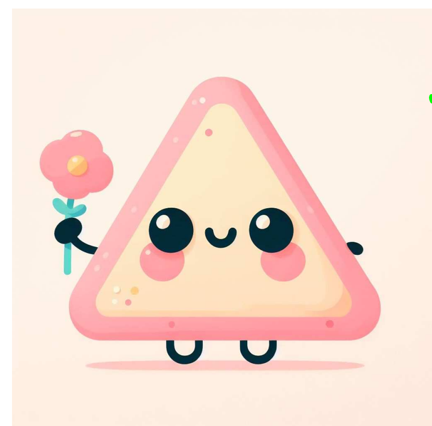


Poor triangle has gas and needs to be burped



Burp lemma If $\overline{AD}$ and $\overline{DE}$ are arcs of leaves and $\measuredangle ADE < \pi$ then there is an arc of a leaf from A to a point on $\overline{DE} \setminus \{D, E\}$.





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- Is there a uniformization theorem for zebra structures?
- Is there a natural topology on the space of zebra structures up to isotopy?
- Can $SL(2,\mathbb{R})$ or $\mathrm{Homeo}_+(\hat{\mathbb{R}})$ be used for renormalization of foliations?
- Are there zebra structures with interesting $\mathrm{Homeo}_+(\hat{\mathbb{R}})$ stabilizers (up to homeomorphism of the surface)?