Hidden symmetries of the dodecahedron

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joint work with:

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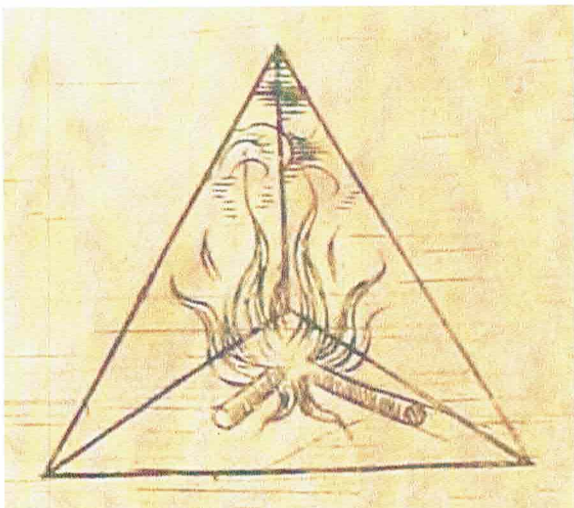
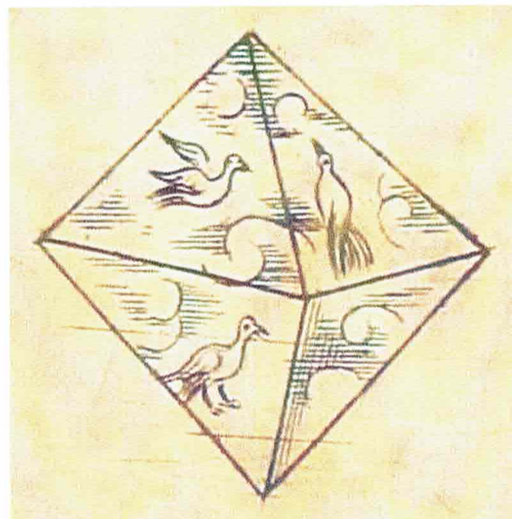
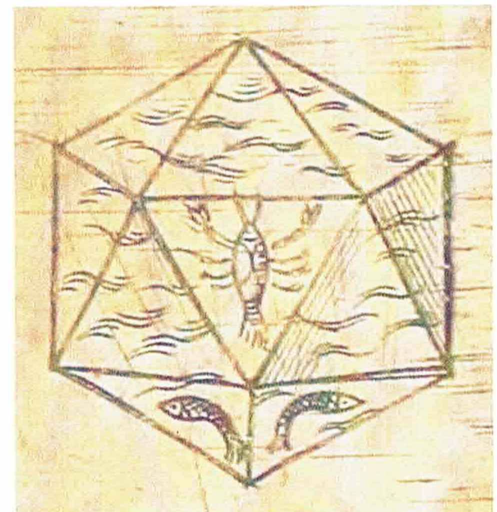
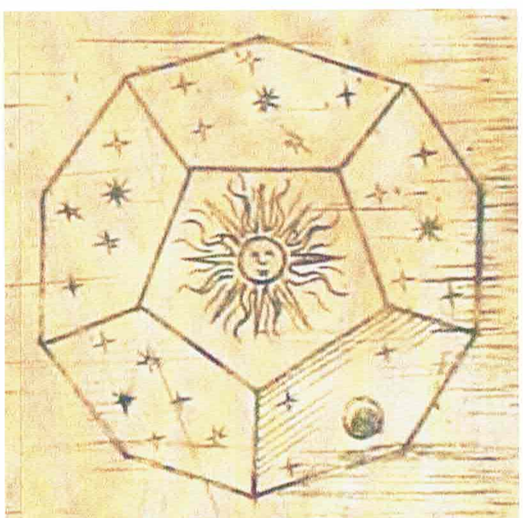
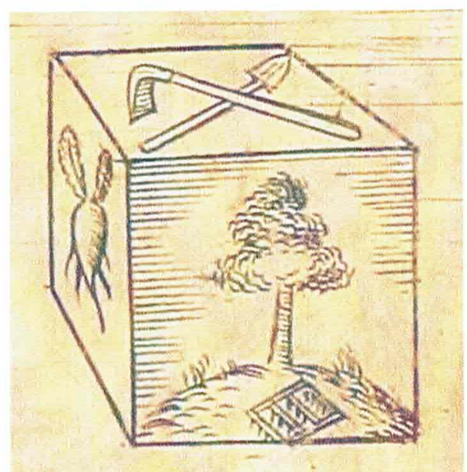
The Platonic solids











Kepler's
Mysterium
Cosmographicum

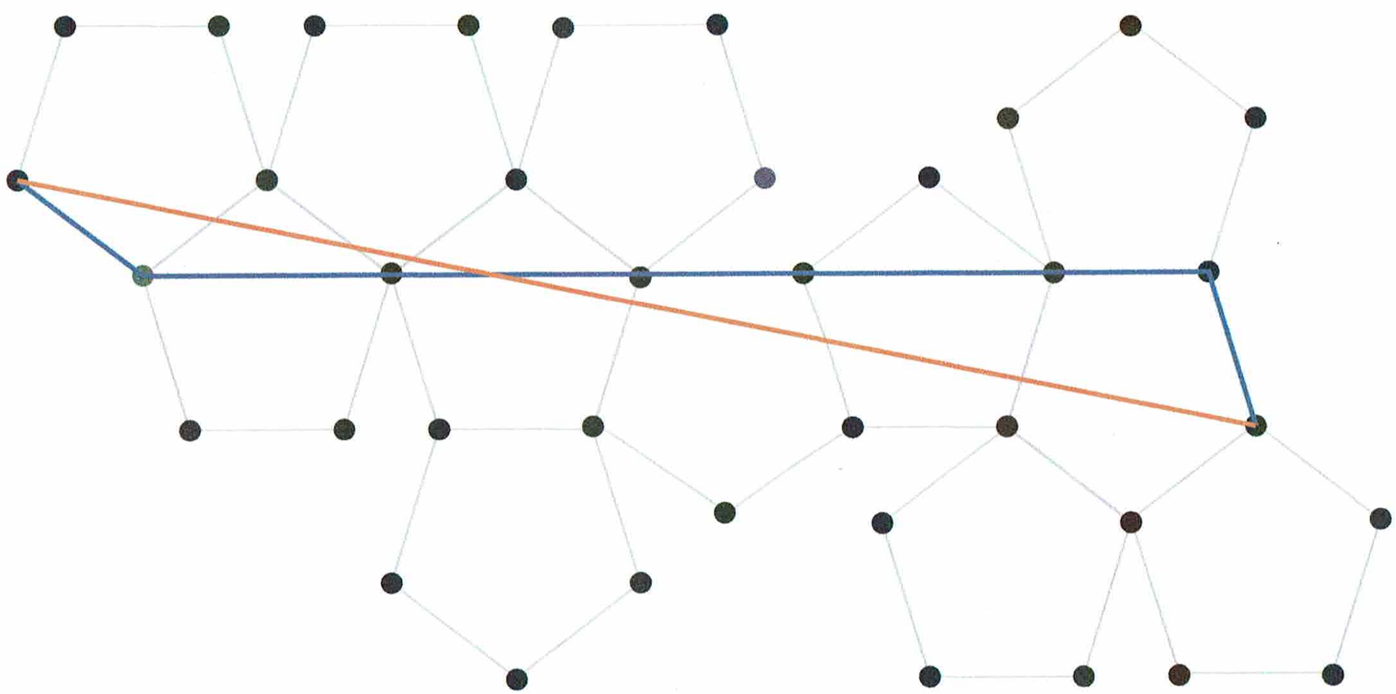
Anti-social jogger Problem

- Jogger starts at a vertex.
- Runs in a straight line on the surface.
- Wants to avoid the other vertices, but return to his home vertex.

On Which of the Platonic solids can the jogger achieve his goals?

A Trajectory from a Vertex to Itself on the Dodecahedron

Jayadev S. Athreya and David Aulicino



Def A <u>saddle</u> <u>connection</u> is a straight-line path joining singularities.

Thm (Petrunin, Athreya-Aulicino) The dodecahedron has a <u>closed</u> saddle connection.

Thm (Davis-Dods-Traub-Yang) The cube does not. (Also, the tetrahedron does not.)

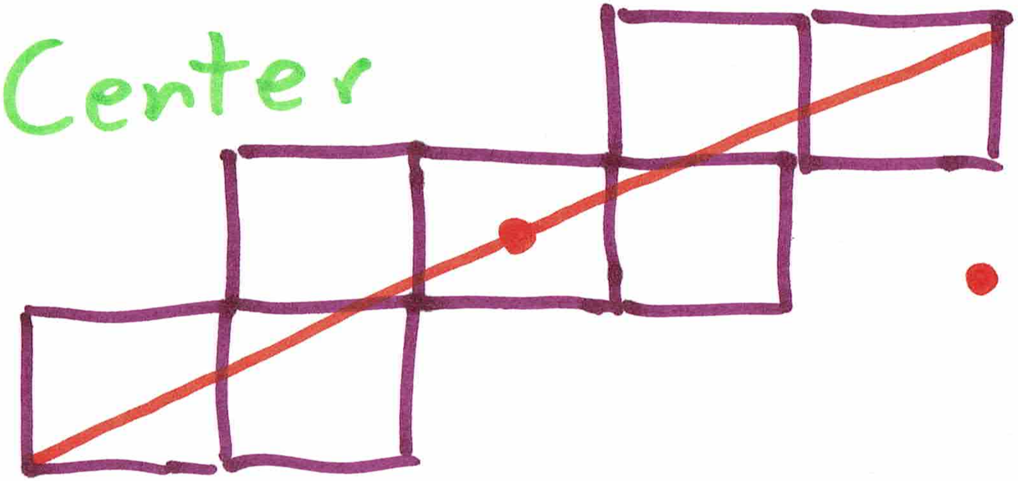
Why not the cube?

1) Midpoints of saddle connections are centers of squares or midpoints of edges.

2) The $180°$ rotation fixing the midpt must swap the endpts of the saddle connection.

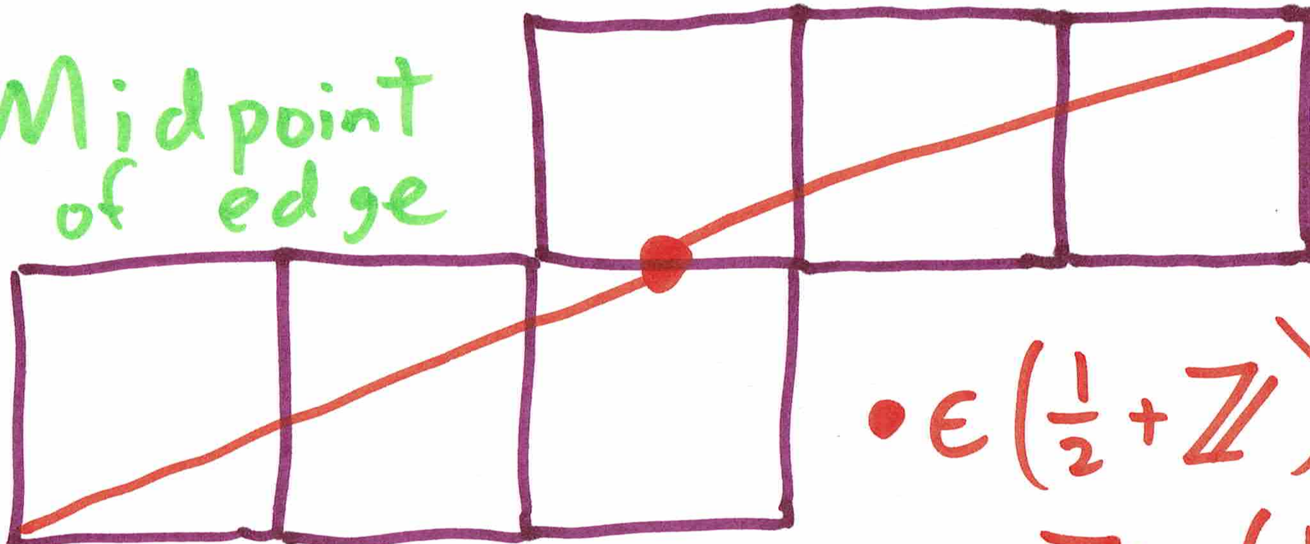
2') The singular endpt of a closed saddle connection must be fixed by the rotation. $\rightarrow\leftarrow$

Center



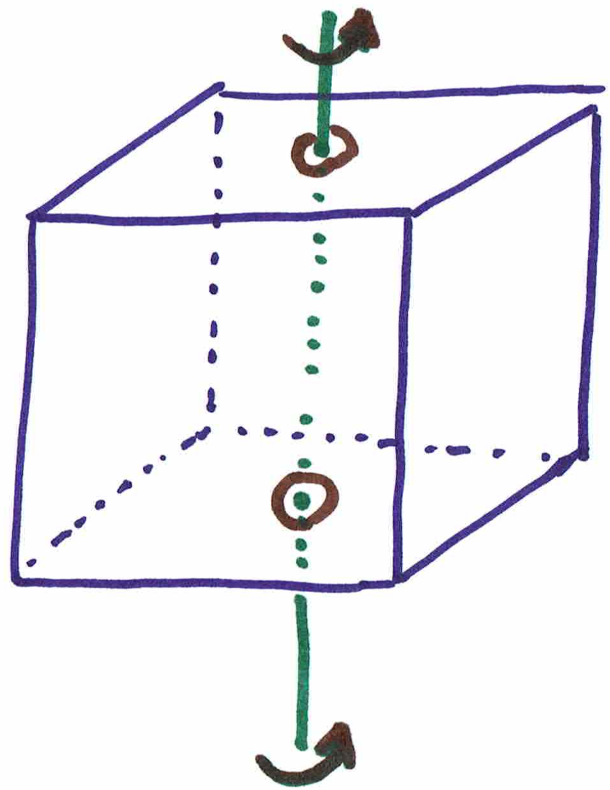
$\bullet \in \left(\frac{1}{2}+\mathbb{Z}\right)^2$

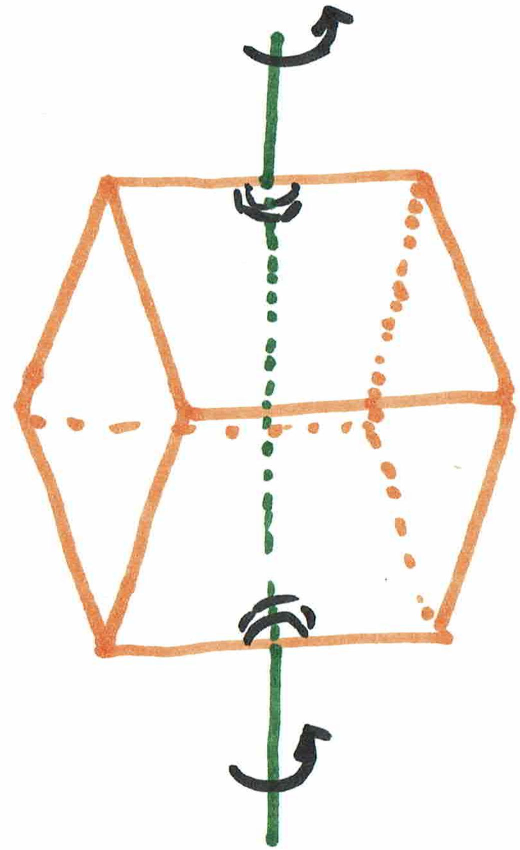
Midpoint of edge



$\bullet \in \left(\frac{1}{2}+\mathbb{Z}\right)\times\mathbb{Z}\ \cup$
$\mathbb{Z}\times\left(\frac{1}{2}+\mathbb{Z}\right).$

Relevant rotations of the cube:





Triangle tiled Platonic solids:

1) Midpoints of saddle connections are also midpoints of edges.

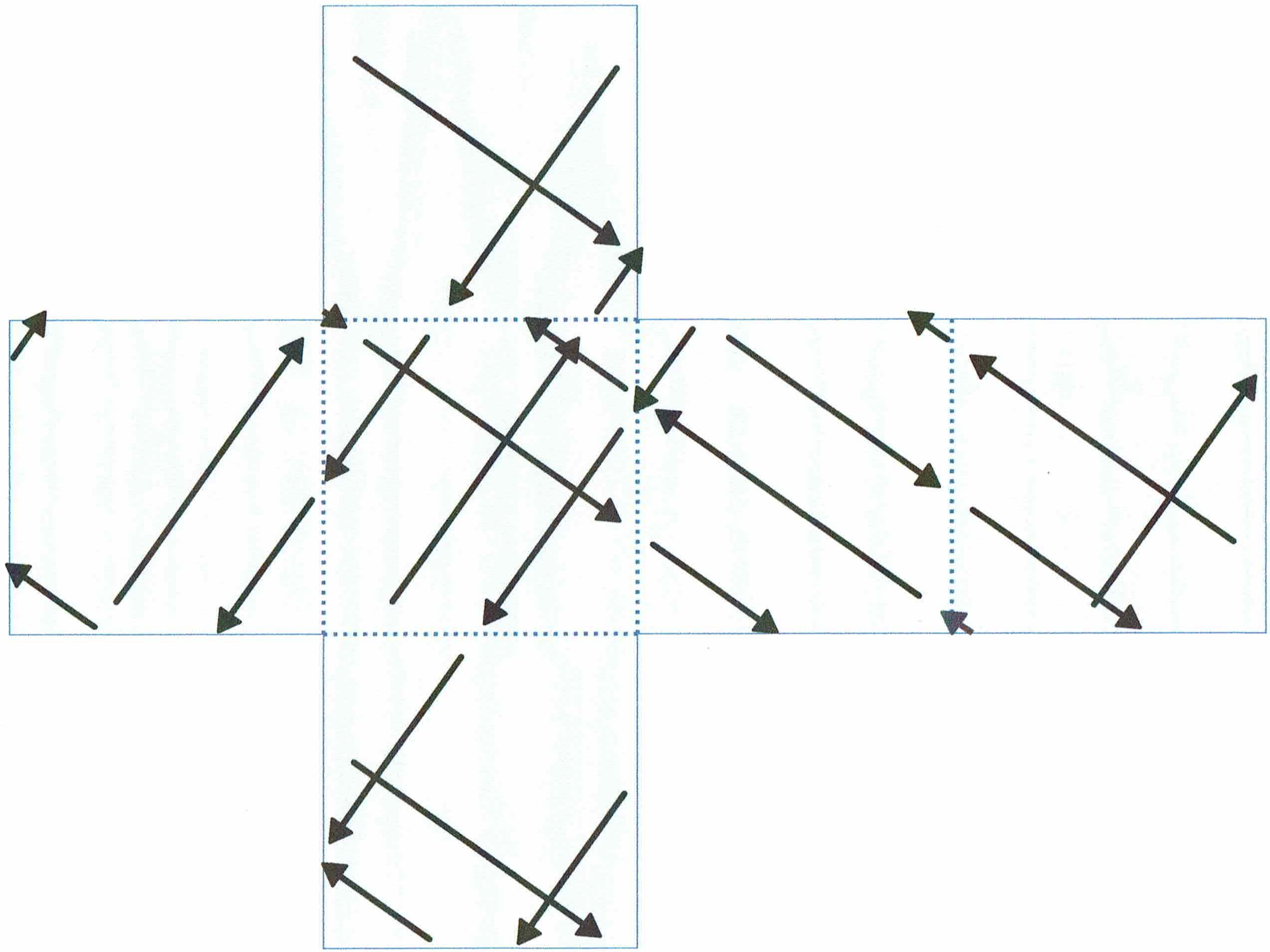
2) The 180° rotation fixing a midpoint of an edge only has one other fixed point: another midpoint of an edge.

Thm (Dmitry Fuchs) There is no closed saddle connection on the octahedron or icosahedron.

Geodesics on rational polyhedra:

Following Fox and Kershner, 1936.

Def A polyhedron (homeomorphic to S^2) is rational if the cone angles all lie in $\pi\mathbb{Q}$.

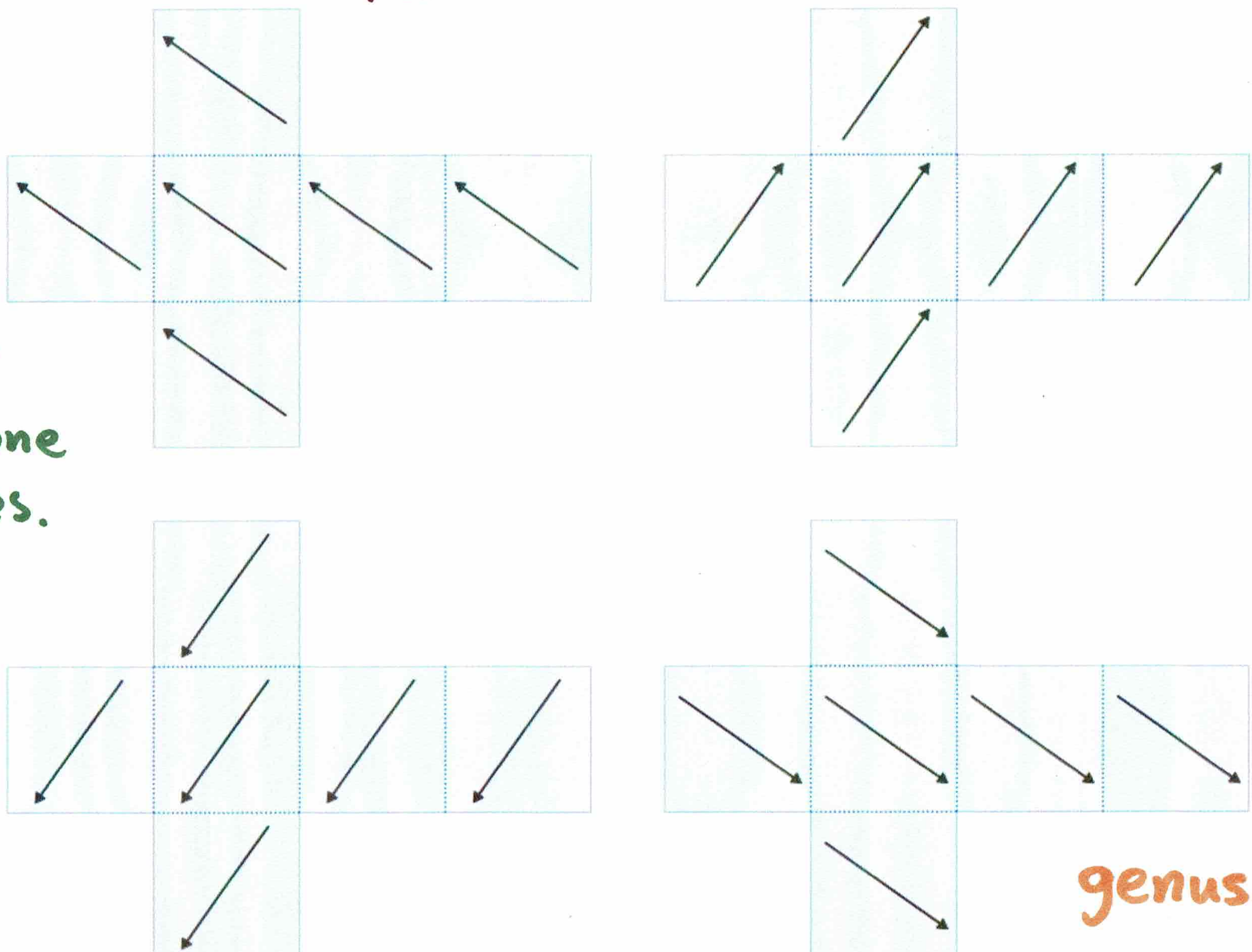


Let S be a rational polyhedron whose cone angles lie in $\frac{2\pi}{k}\mathbb{Z}$ for some integer $k \geq 1$. (Take k minimal.)

Let S° be S with its singularities removed. Let U be the unit tangent bundle of S°.

Prop The map obtained using a net $\mathrm{dir}_k : U \to \mathbb{R} / \frac{2\pi}{k}\mathbb{Z}$ is geodesic flow & parallel transport invariant.

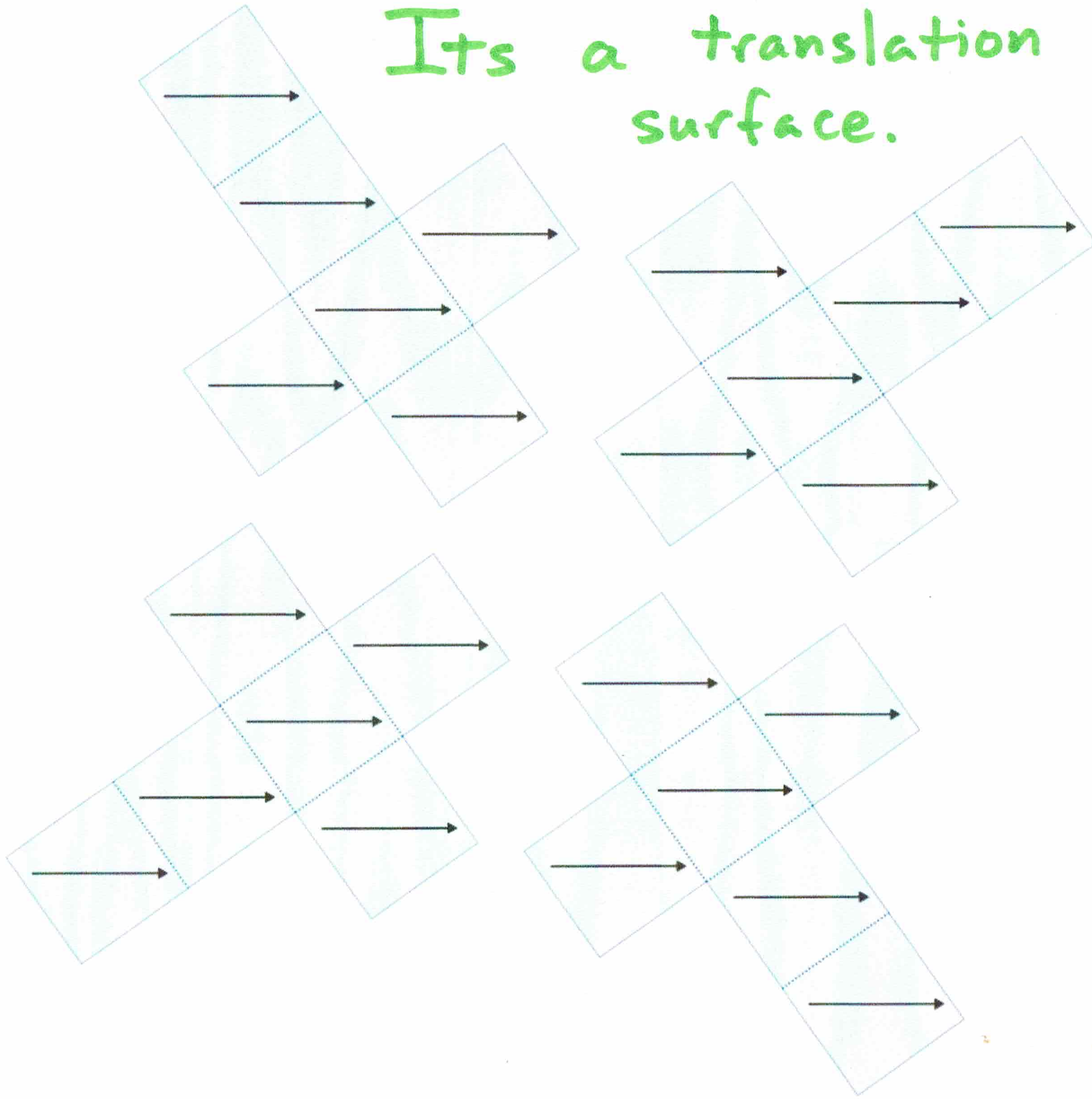
The fibers $\mathrm{dir}_K^{-1}\left(\theta + \frac{2\pi}{K}\mathbb{Z}\right)$ are all isometric to the same singular flat surface.

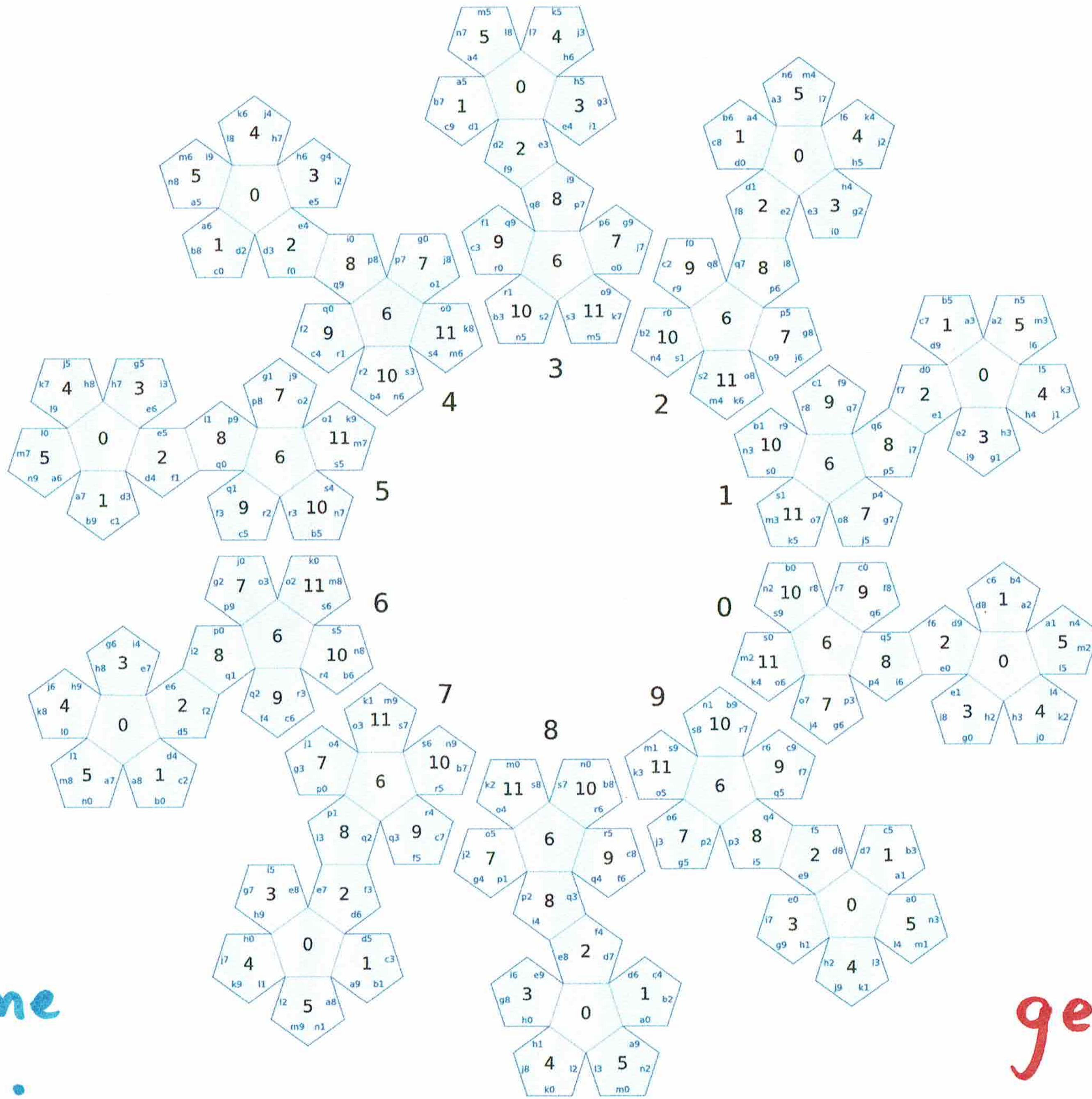


Eight 6π cone angles.

genus 9

Its a translation surface.



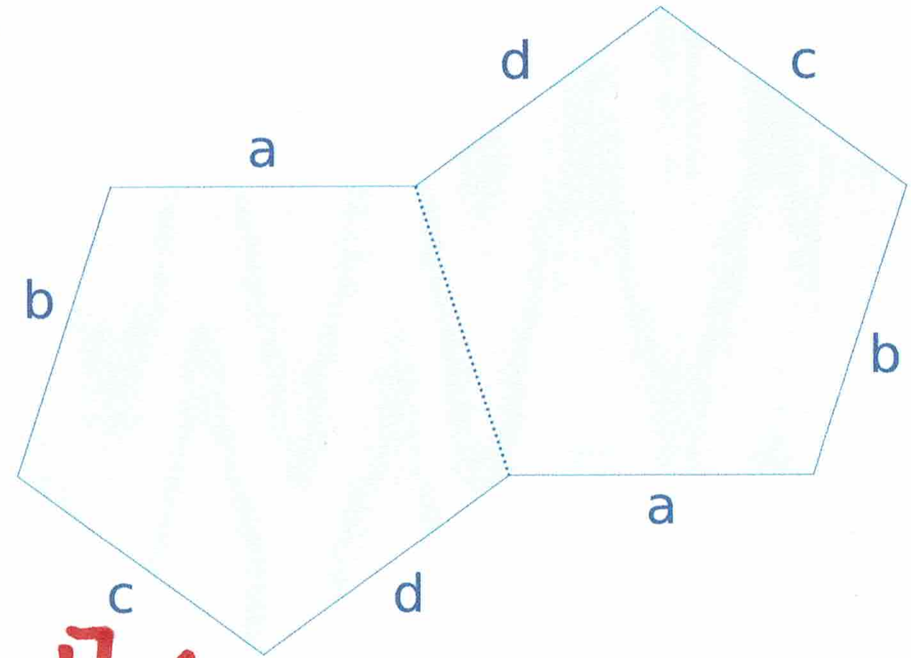


Twenty 18π cone angles.

genus 81

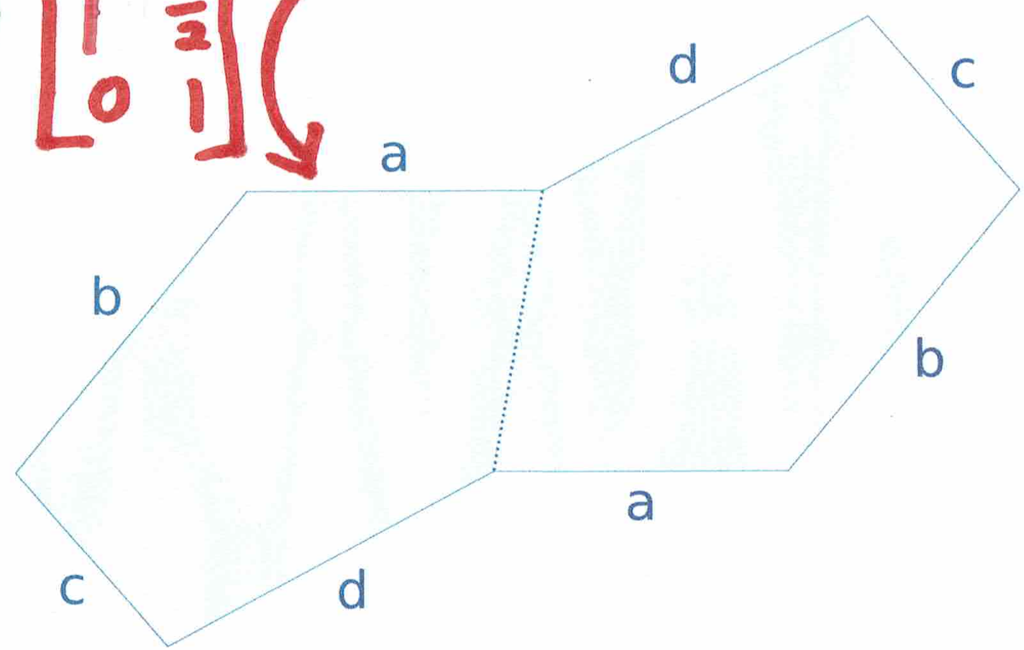
Symmetries of Translation Surfaces

A *translation surface* is formed by gluing together polygons by translation.



The group $SL(2,\mathbb{R})$ acts.

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$



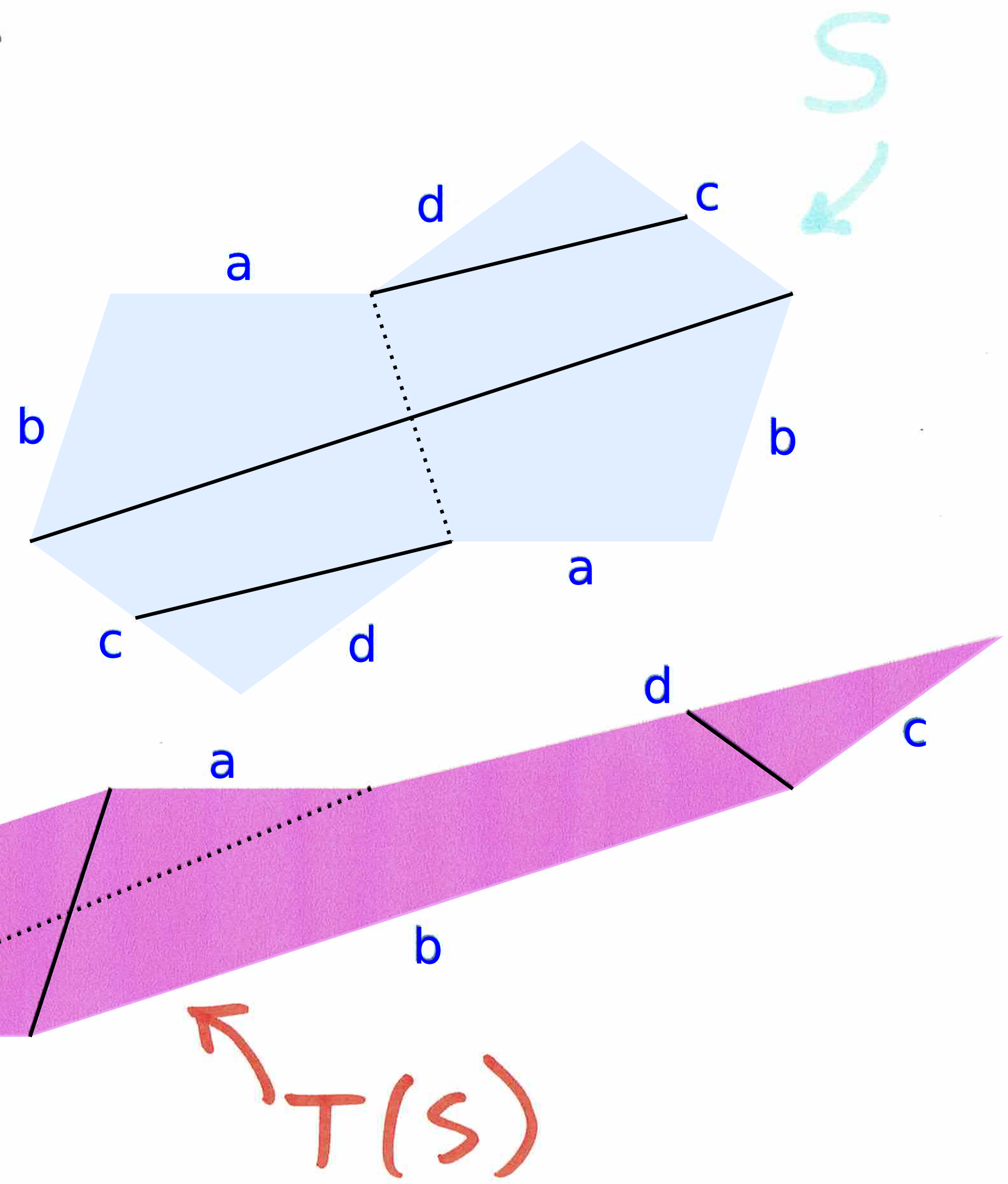
The Veech group of a translation surface S is the stabilizer

$$\Gamma S \subset SL(2,\mathbb{R}).$$

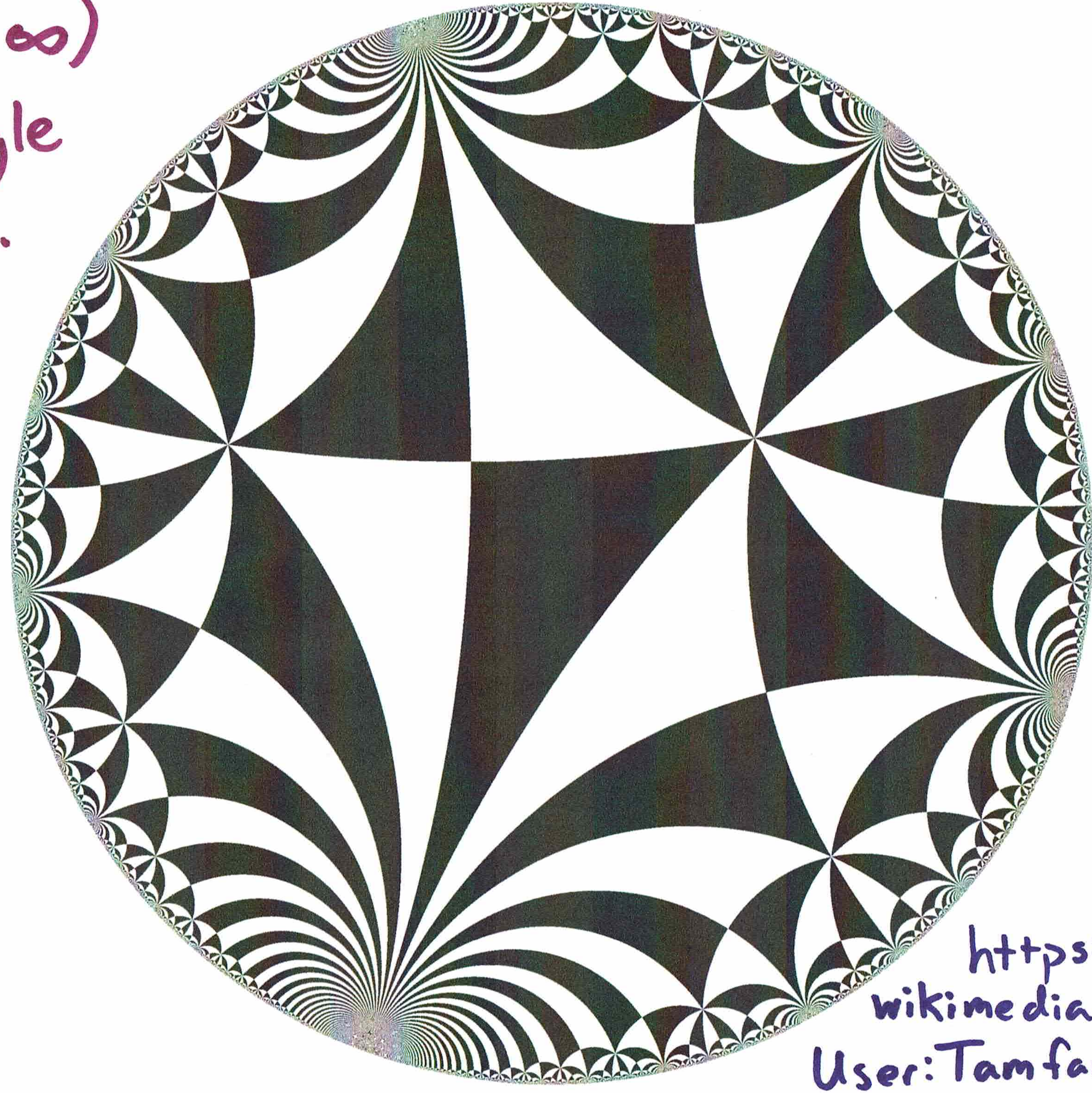
Examples:

$$R = \begin{bmatrix} \cos\frac{\pi}{5} & -\sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & \cos\frac{\pi}{5} \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 2.752\ldots \\ 0 & 1 \end{bmatrix}$$



$(2,5,\infty)$ triangle group.



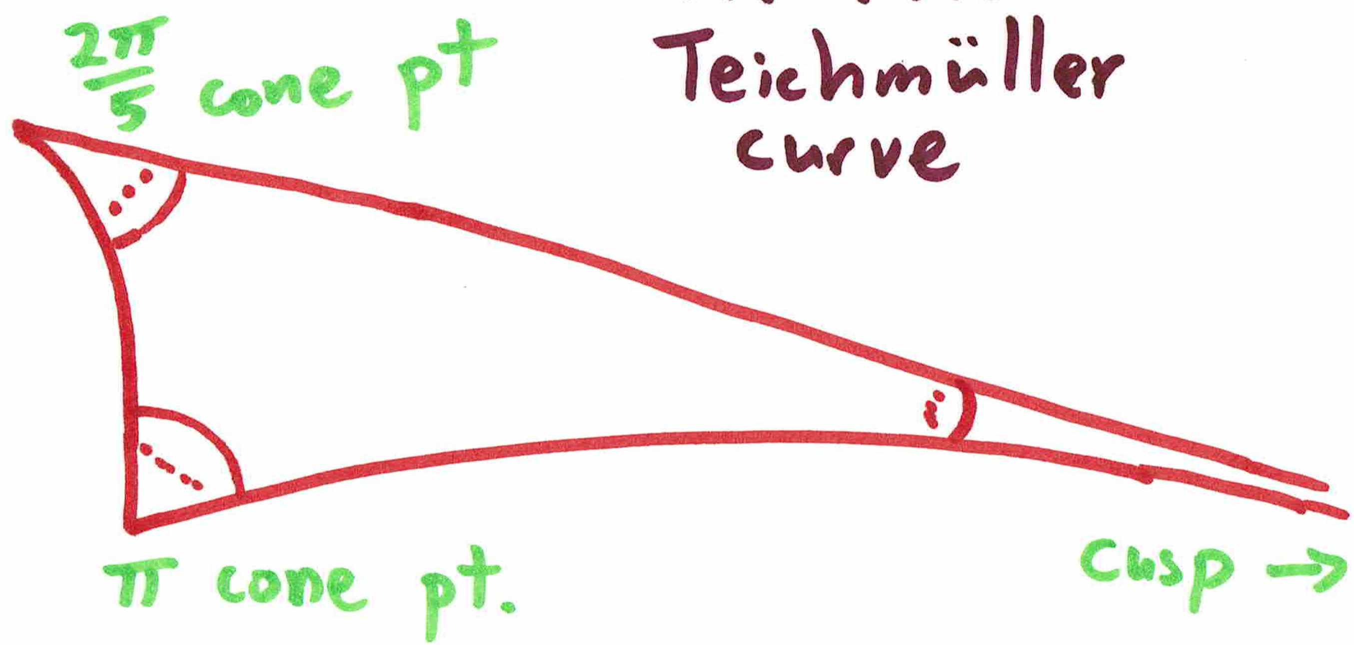
https://commons.wikimedia.org/wiki/User:Tamfang/H2

Affine images of S of the same area are parameterized by

$$SL(2,\mathbb{R})/\Gamma S.$$

Up to rotation they are parameterized by $\mathbb{H}^2/\Gamma S$.

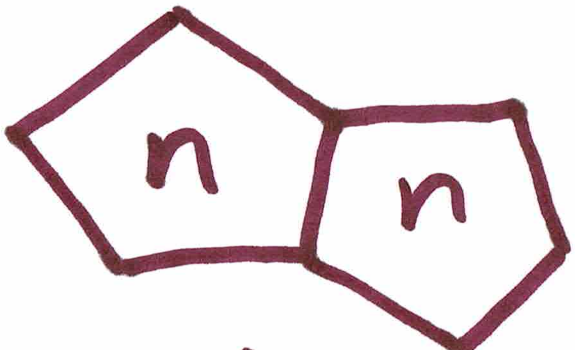
Teichmüller curve



A translation surface has the **lattice property** if $\mathbb{H}^2/\Gamma S$ is finite volume.

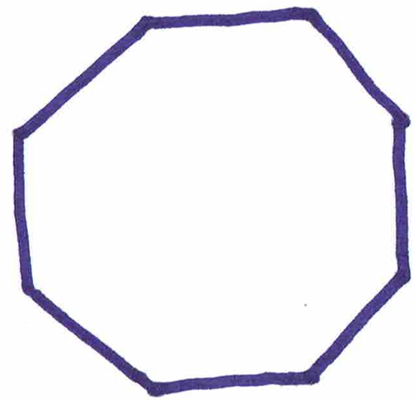
Veech Examples:

n odd



$\Gamma S = \Delta(2, n, \infty)$

n even



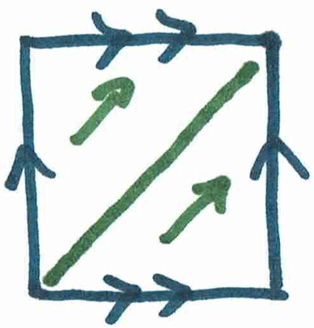
$\Gamma S = \left(\frac{n}{2}, \infty, \infty\right)$

(Actually ΓS is bigger if $n \leq 6$.)

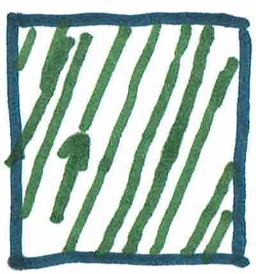
Thm (Veech dichotomy)

In any direction on a surface with the lattice property either

(1) The surface decomposes into parallel cylinders and saddle connections (<u>com</u>p<u>le</u>te <u>periodic</u>ity)



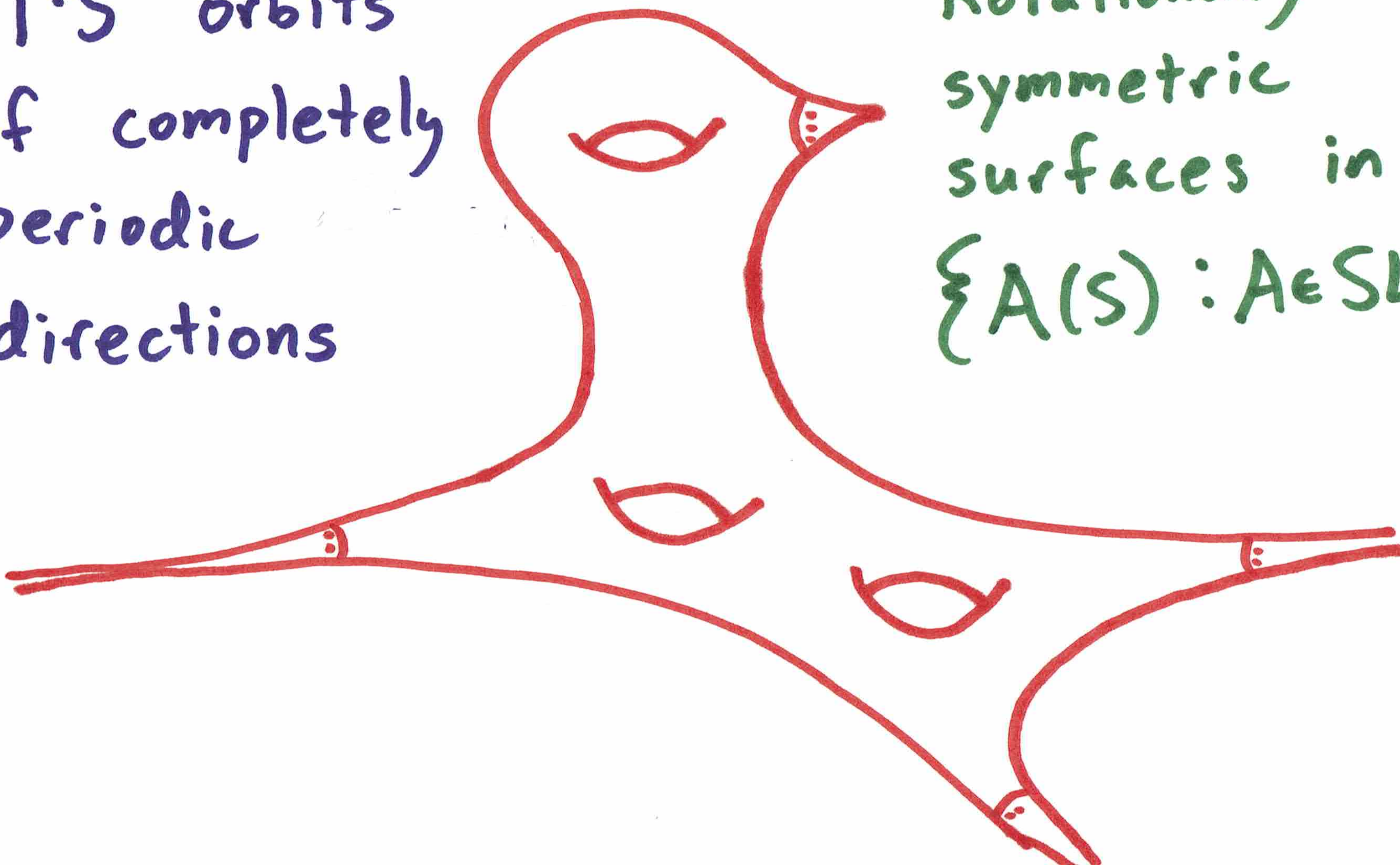
or (2) The foliation is uniquely ergodic. (All leaves equidistribute.)



Teichmüller curves $\mathbb{H}^2/\Gamma S$.

__Cusps__:
ΓS orbits of completely periodic directions

__Cone points__:
Rotationally symmetric surfaces in $\{A(S) : A \in SL_2\mathbb{R}\}$.



Prop: A finite cover of a lattice surface (possibly branched over the singularities) also has the lattice property.

Idea of proof: Consider $\tilde{S} \to S$.

① If $A \in \Gamma S$, then $A(\tilde{S})$ is another cover of the same degree.

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{A} & A(\tilde{S}) \\ \downarrow & & \downarrow \\ S & \xrightarrow{A} & A(S) = S \end{array}$$

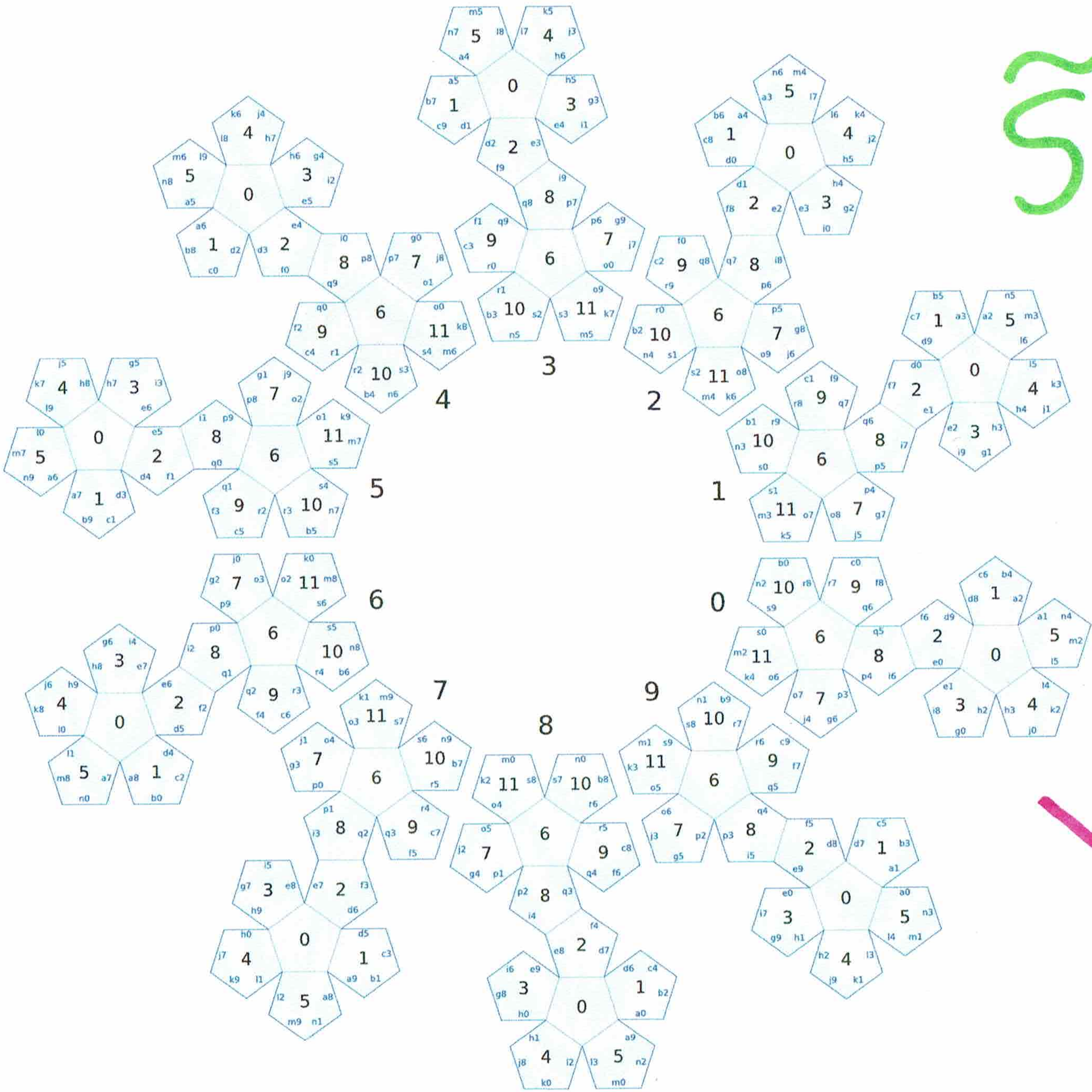
② There are only finitely many covers of fixed degree.

$$\Gamma S \xrightarrow[\varphi]{} \mathrm{Perm}(\mathrm{Covers})$$

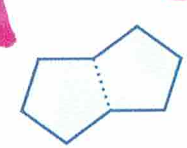
$$\Gamma \tilde{S} = \varphi^{-1}(\mathrm{Stab}\ \tilde{S})$$

The dodecahedron

$\tilde{S}$



covering of degree 60



S

Facts about $\Gamma\tilde{S}$ (Athreya-Aulicino-H)

1) $\Gamma\tilde{S}$ is index 2106 in ΓS.
2) $\mathbb{H}^2/\Gamma\tilde{S}$
   (a) has genus 131,
   (b) has 362 cusps,
   (c) has 18 cone singularities with cone angle π
   (d) has a single cone point with angle $\frac{2\pi}{5}$ (representing $\tilde{S}$).

Q: Is there a good way to visualize $\mathbb{H}^2/\Gamma\tilde{S}$?

Facts about geodesics on the dodecahedron

Up to symmetry (Affine symmetries of $\tilde{S}$) there are:

(a) 422 maximal immersed cylinders,

(b) 422 saddle connections,

(c) 31 closed saddle connections.

Rk There are 211 cusps of $\mathbb{H}/\Gamma_{\pm}\tilde{S}$ (i.e. allowing orientation reversing symmetries).