Workshop on Dynamical Systems and Related Topics
In memory of Bill Veech
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# Dynamics of Pseudo-Anosovs on a limit of Veech's surfaces

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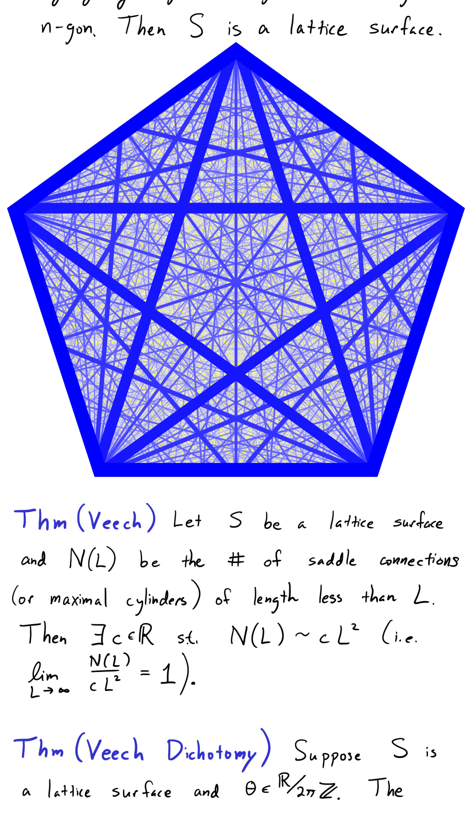
**Def** Let $S$ be a (half-)translation surface. The **Veech group** $\Gamma$ of $S$ is the stabilizer of $S$ in $SL(2,\mathbb{R})$.

**Def** $S$ has **Veech's lattice property** (is a **lattice surface**) if $\Gamma \subset SL(2,\mathbb{R})$ is a lattice, i.e. $Vol(SL(2,\mathbb{R})/\Gamma) < \infty$.

**Thm (Veech '89)** Fix $n \geq 3$ and let $S$ be a (half-) translation surface built by gluing together edges of a regular $n$-gon. Then $S$ is a lattice surface.



**Thm (Veech)** Let $S$ be a lattice surface and $N(L)$ be the # of saddle connections (or maximal cylinders) of length less than $L$. Then $\exists c \in \mathbb{R}$ s.t. $N(L) \sim cL^2$ (i.e. $\lim_{L\to\infty} \frac{N(L)}{cL^2} = 1$).

**Thm (Veech Dichotomy)** Suppose $S$ is a lattice surface and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. The straight-line flow in direction $\theta$ is either completely periodic or uniquely ergodic.

## Infinite Lattice Surfaces

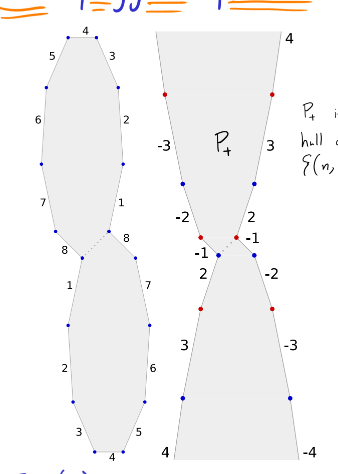
**Question** Is there a finite area infinite genus lattice surface?

**Covers** A $G$-(branched) cover of $S$ is a cover $\tilde{S} \to S$ with deck group isomorphic to $G$.

**Thm (Frączek-Schmoll, 2017)** If $\tilde{S}$ is a lattice surface and a $\mathbb{Z}$-cover of a closed translation surface then the straight line flow on $\tilde{S}$ is ergodic in a.e. direction.

**Branched cover case:** Partial results by Ralston-Troubetzkoy, Hubert-Weiss, ...

## The polygonal parabola surface.




$P_+$ is the convex hull of $\{(n, n^2) : n \in \mathbb{Z}\}$.

**Thm (H.)** The polygonal parabola surface $P$ is a lattice surface:

$$\Gamma_2 = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\rangle \text{ stabilizes } P.$$

**Obs.** The # of cylinders / saddle connections of length $<L$ grows quadratically.

**Today's Question:** How do hyperbolic elements of $\Gamma_2$ act on $P$?

> Echos of Veech's surfaces dominate dynamics on the $P$, the polygonal parabola surface

## Pseudo-Anosovs (following Thurston)

**Thm (Thurston)** If $[\varphi] \in MCG(S)$ is not finite order and does not fix a simple closed curve, then it is pseudo-Anosov, ie. $\exists$ a half translation structure on $S$ and an affine $\varphi : S \circlearrowleft$ s.t. $\varphi \in [\varphi]$ and $D\varphi = \begin{bmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{bmatrix}$ for $\lambda > 1$.

**Furthermore:** ① $\varphi$ is mixing.

② The horizontal and vertical flows are uniquely ergodic.

③ Let $\mathcal{S} = \{$simple closed curves on $S\}$
Geometric intersection #: $i : \mathcal{S} \times \mathcal{S} \to \mathbb{Z}_{\geq 0}$

**Thm** $\forall \alpha, \beta \in \mathcal{S}$, $i(\varphi^n \alpha, \beta) \sim \lambda^n \nu^s(\alpha)\nu^u(\beta)$ where $\nu^s$ and $\nu^u$ are the Lebesgue transverse measures to the stable and unstable foliations of $\varphi$.

## Returning to the Polygonal Parabola.

**Thm 1 (H.)** Let $\varphi : P \circlearrowleft$ be a hyperbolic affine homeomorphism. Then $\exists c_\varphi > 0$ s.t $\forall$ cylinders $\mathcal{A}$ & $\mathcal{B}$,

$$Area(\varphi^n(\mathcal{A}) \cap \mathcal{B}) \sim \frac{1}{4\sqrt{2\pi}}(c_\varphi n)^{-\frac{3}{2}} Area(\mathcal{A}) Area(\mathcal{B})$$

**Cor** $\varphi : P \circlearrowleft$ is not recurrent.

**Thm 2:** Fix $\varphi : P \circlearrowleft$ hyperbolic. Then $\forall \alpha, \beta \in \mathcal{S}$,

$$i(\alpha, \beta) \underset{n\to\infty}{\sim} \frac{\lambda^n \nu^s(\alpha)\nu^u(\beta)}{(nc_\varphi)^{\frac{3}{2}} 4\sqrt{2\pi}\, |u^s \wedge u^u|}.$$

Geometric intersection number.
(unit vectors in stable/unstable directions)

**Rk:** Similar statements hold for saddle connections.

## Homology

Let $\Sigma$ be the singularities of $P$.

$H_1(P \setminus \Sigma; \mathbb{R}) = \{$homology classes of finite $\mathbb{R}$-weighted sums of closed curves$\}$

$H_1(P, \Sigma; \mathbb{R}) = \{$'...' of arcs joining singularities$\}$

Algebraic intersection # is a bilinear map
$\cap : H_1(P \setminus \Sigma; \mathbb{R}) \times H_1(P, \Sigma; \mathbb{R}) \to \mathbb{R}$.

**Thm (H)** $\forall \varphi : P \circlearrowleft$ hyp. there is a filtration $H_1(P \setminus \Sigma; \mathbb{R}) = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots$ s.t.

① $\mathcal{F}_{i+1}$ is $\mathbb{R}$-codim 1 in $\mathcal{F}_i$, ② $\bigcap_i \mathcal{F}_i = \{0\}$

③ $\forall \alpha \in \mathcal{F}_j \setminus \mathcal{F}_{j+1}$ $\exists C \neq 0$ s.t $\beta \in H_1(S, \Sigma; \mathbb{R})$,

$$\lim_{n\to\infty} \frac{n^{j+3/2}}{\lambda^n}(\varphi^n(\alpha) \cap \beta) = C\,\nu^u(\beta).$$

$$\left(\text{If } j = 0,\ C = \frac{\nu^s(\alpha)}{c_\varphi^{3/2}\, 4\sqrt{2\pi}\, |u^s \wedge u^u|}.\right)$$

## Echo's of Veech's surfaces.

Let $c = \cos\frac{2\pi}{n}$.

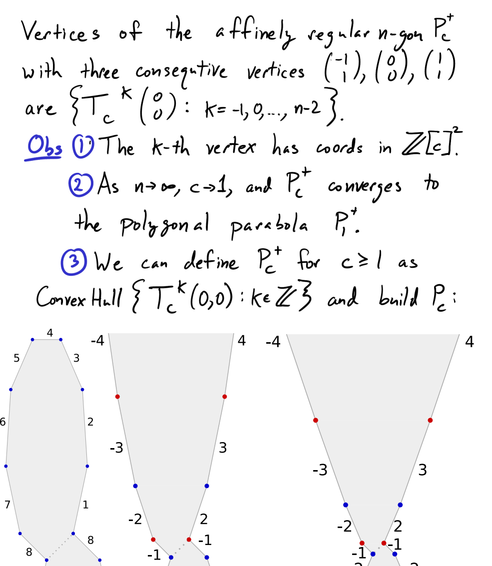
Let $T_c : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} c & c-1 \\ c+1 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Vertices of the affinely regular $n$-gon $P_c^+$ with three consequtive vertices $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are $\{T_c^k \begin{pmatrix} 0 \\ 0 \end{pmatrix} : k = -1, 0, \ldots, n-2\}$.

**Obs** ① The $k$-th vertex has coords in $\mathbb{Z}[c]^2$.

② As $n \to \infty$, $c \to 1$, and $P_c^+$ converges to the polygonal parabola $P_1^+$.

③ We can define $P_c^+$ for $c \geq 1$ as Convex Hull $\{T_c^k(0,0) : k \in \mathbb{Z}\}$ and build $P_c$:




**Def** $\widetilde{hol} : H_1(P_1, \Sigma; \mathbb{R}) \to \mathbb{R}[c]^2$
$\alpha \longmapsto (c \mapsto hol_c\, \alpha)$
holonomy on $P_c$ for $c \geq 1$.

**Thm** The Veech group of $P_c$ for $c \geq 1$ is $\left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c & c-1 \\ c+1 & c \end{bmatrix} \right\rangle$.

**Def** $\rho : F_2 \to SL(2, \mathbb{R}[c])$ is as above.

**Def** If $\varphi : P \circlearrowleft$ is hyperbolic then $D\varphi = \rho(g)(c=1)$ and $c_\varphi := \frac{d}{dc}\left[\log \lambda_c\right]_{c=1}$ where $\lambda_c > 1$ is the unstable eigenvalue of $\rho(g)(c)$.

**Main Lemma** $\forall \alpha \in H_1(P_1 \setminus \Sigma; \mathbb{R})$, $\beta \in H_1(P_1, \Sigma; \mathbb{R})$

$$\alpha \cap \beta = \frac{1}{2\pi}\int_0^\pi \left[(\widetilde{hol}\, \alpha \wedge \widetilde{hol}\, \beta)(\cos\theta)\right](1 - \cos\theta)\, d\theta$$

**Cor** $$(\varphi^n \alpha) \cap \beta = \frac{1}{2\pi}\int_0^\pi \left[(\rho_c(g) \cdot hol_c\, \alpha)^n \wedge hol_c\, \beta\right](1-c)\, d\theta$$

**Erdélyi's Thm:** Given $p(t)$ increasing and $q(t)$ both analytic, gives an asymptotic expansion of $\int_0^\varepsilon e^{-np(t)} q(t)\, dt$ in terms of the power series of $p$ & $q$.