Workshop on Dynamics and Moduli Spaces of Translation Surfaces

October 22 - 26, 2018, The Fields Institute

Infinite nilpotent covers of square-tiled surfaces

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The Extrinsic Primitive Torsion Problem arXiv:1708.02093

Outline of talk

1. Definitions, basic examples, motivating results
2. Symmetric square tiled surfaces through cylinders

Definitions

- Let G be a discrete group and S be a surface. A G-*cover* of S is a regular cover $\tilde{S} \to S$ with Deck group isomorphic to G.
- Let μ be a measure on a G-cover $\tilde{S}$, and $\alpha : G \to \mathbb{R}_{>0}$ be a group homomorphism. Then μ is *Maharam* if

$$\mu \circ g = \alpha(g)\mu \quad \text{for all } g \in G.$$

Theorem (Babillot, Ledrappier–Sarig)
Let $\tilde{S}$ is a nilpotent-cover of a compact hyperbolic surface and $h^t : T_1\tilde{S} \to T_1\tilde{S}$ be the horocycle flow. Then the the ergodic horocycle-flow invariant Radon measures are the Maharam measures which are in bijective correspondence with $Hom(G, \mathbb{R}_{>0}) = \{\alpha : G \to \mathbb{R}_{>0}\}$.

- A square-tiled surface is a cover of $\mathbb{T}^* = \mathbb{T}^2 \setminus \{0\}$.
- The straight-line flow in direction θ is

$$F^t_\theta : S \to S;\ z \mapsto z + e^{i\theta} \text{ in local coordinates.}$$

Theorem (H) Let $\tilde{S}$ be a nilpotent-cover of a compact translation surface S which is possibly branched. Assume there are horizontal and vertical affine multitwists $\phi, \psi : \tilde{S} \to \tilde{S}$ in horizontal and vertical cylinder decompositions and that each cylinder intersects at least two others. Let

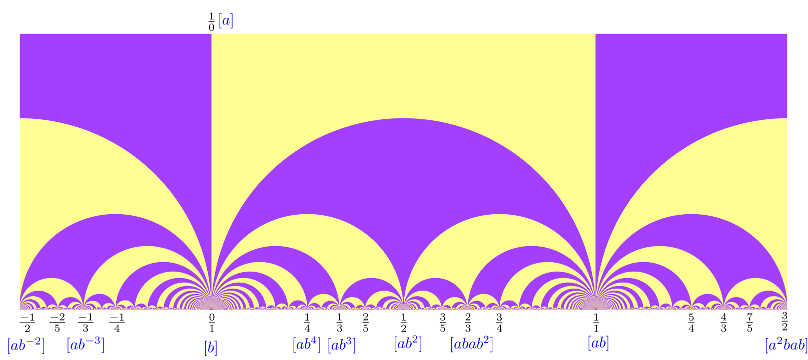
$$\Gamma = \langle D(\phi), D(\psi) \rangle = \left\langle \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \right\rangle .$$

Then for all but countably many θ in the limit set of Γ, the flow F_θ is ergodic. Moreover, the locally-finite ergodic invariant Borel measures are the α-Maharam measures and there is one for each $\alpha : G \to \mathbb{R}_{>0}$.

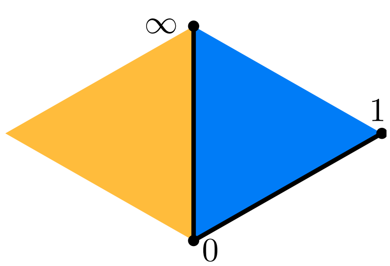
Symmetric square tiled surfaces through cylinders

Def. Let S be a square-tiled surface with covering map $\pi : S \to \mathbb{T}^*$. Let $k \geq 2$.} We say S is k-periodic if for every non-singular geodesic $\tilde{\gamma}$ of rational slope in S, the restriction of π to $\tilde{\gamma} \to \pi(\tilde{\gamma})$ is a finite cover of degree dividing k.

Prop. For each integer $k \geq 2$, there is a universal k-periodic square-tiled surface U_k. That is, U_k is a k-periodic square tiled surface and if S is another one, there is a covering $U_k \to S$.

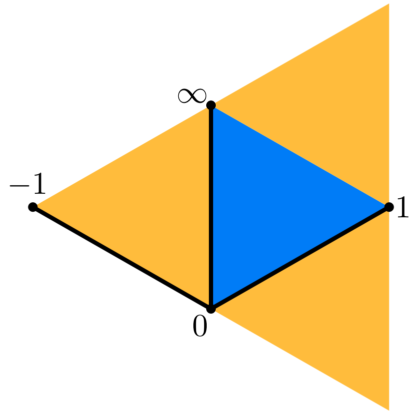


Normal generators for P_2.



Vertex	Generator of P_2
∞	a^2
0	b^2
1	$(ab)^2$

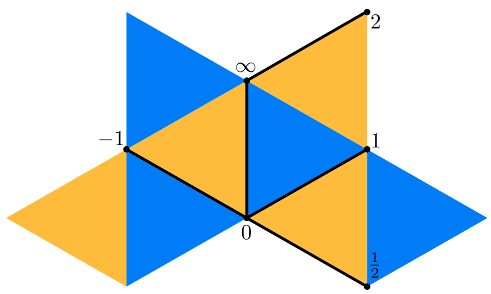
Normal generators for P_3.



Vertex	Generator of P_3
∞	a^3
0	b^3
1	$(ab)^3$
-1	$(ab^{-1})^3$

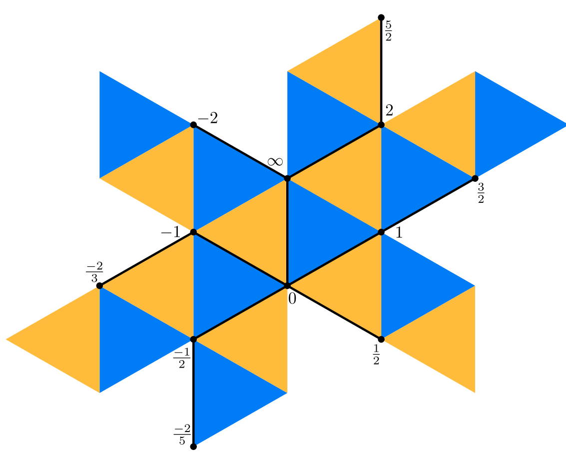
$$G_3 = F_2/P_3 = F_2/\langle a^3, b^3, (ab)^3, (ab^{-1})^3 \rangle.$$

Normal generators for P_4.



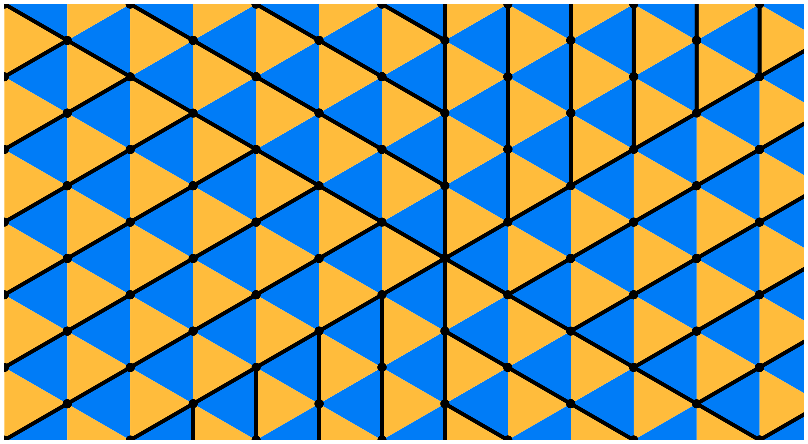
Vertex	Generator of P_4
∞	a^4
0	b^4
1	$(ab)^4$
-1	$(ab^{-1})^4$
2	$(a^2b)^4$
$\frac{1}{2}$	$(ab^2)^4$

Normal generators for P_5.



Vertex	Generator of P_5
∞	a^5
0	b^5
1	$(ab)^5$
-1	$(ab^{-1})^5$
2	$(a^2b)^5$
$\frac{1}{2}$	$(ab^2)^5$
-2	$(a^2b^{-1})^5$
$\frac{-1}{2}$	$(ab^{-2})^5$
$\frac{3}{2}$	$(a^2bab)^5$
$\frac{-2}{3}$	$(ab^{-1}ab^{-2})^5$
$\frac{5}{2}$	$(a^3ba^2b)^5$
$\frac{-2}{5}$	$(ab^{-2}ab^{-3})^5$

Normal generators for P_6...



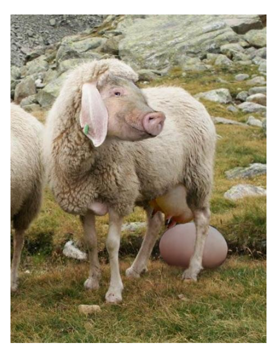
Open Question: Are the groups P_k for $k \geq 6$ finitely normally generated? Is our countable list of generators minimal?

Theorem 1 (H – Bou-Rabee).
The surface U_k is infinite (equivalently, F_2/P_k is infinite) if and only if $k \geq 4$.

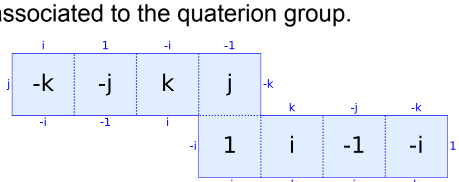
Theorem 2 (H – Bou-Rabee).
The group $G_4 = F_2/P_4$ is virtually a torsion-free nilpotent group of dimension 5 and nilpotence class 2. There is a faithful representation $\tilde{\tilde{\rho}} : F_2/P_4 \to SL(9, \mathbb{C})$ determined by

$$\tilde{\tilde{\rho}}(a) = diag(1, -1, -i, -i; -1, 1, i, i; 1),$$

$$\tilde{\tilde{\rho}}_4(b) = \left(\begin{array}{cccc|cccc|c} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$



Cor. U_4 is a torsion-free class 2 nilpotent cover of the *eierlegende Wollmilchsau*, the 8-square surface built associated to the quaterion group.



This surface was found independently by Forni and by Herrlich – Möller – Weitze-Schmithüsen.

The surface U_4 is a $\mathbb{Z}$-cover of a $\mathbb{Z}^4$-cover of the Wollmilchsau. The deck group $G_4 = F_2/P_4$ is a $\mathbb{Z}$-extension of a $\mathbb{Z}^4$-extension of the quaternion group.

Theorem (Frączek – Schmoll)
Straight-line flow is ergodic in a.e. direction on $U_4/\mathbb{Z}$.

There is a decomposition $Aut(F_2) = Aut_+(F_2) \cup Aut_-(F_2)$, where signs are assigned to an automorphism depending on the determinant of the image in $GL(2, \mathbb{Z}) = Out(F_2)$.

Def. A representation $\rho : F_2 \to GL(m, \mathbb{C})$ is *(oriented) characteristic* if:

1. For every $\psi \in Aut_+(F_2)$ there is an $M \in GL(m, \mathbb{C})$ so that

$$M \cdot \rho \circ \psi^{-1}(g) \cdot M^{-1} = \rho(g) \quad \text{for all } g \in F_2.$$

2. For every $\psi \in Aut_-(F_2)$ there is an $M \in GL(m, \mathbb{C})$ so that

$$M \cdot \overline{\rho \circ \psi^{-1}(g)} \cdot M^{-1} = \rho(g) \quad \text{for all } g \in F_2.$$