Immersions and the space of all translation surfaces

Pat Hooper
(City College of New York + CUNY Graduate Center)

Outline of talk:

I. Examples of convergence
II. Definition of topologies
III. Examples illustrating their use
IV. Fundamental results about the topology.

I. Examples of convergence

Example (Bowman) The Arnoux-Yoccoz "converge" as $g \to \infty$.

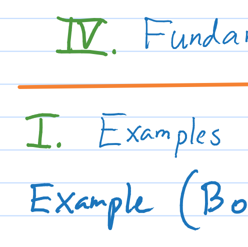

FIGURE 6. Outlines of the surfaces (X_g, ω_g) for $g = 3, 4, 5, 6$.

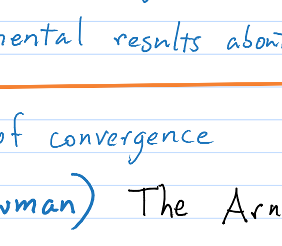

FIGURE 5. The surface $(X_\infty, \omega_\infty)$.

II. Definition of topologies

Def A translation structure on a topological surface is an atlas of charts (local homeo.) to the plane so that the transition functions are local translations.

Translation structures have no singularities!

Immersions, Embeddings, Isomorphisms

Let (R, o_R) and (S, o_S) be pointed translation surfaces and let $A \subset R$ and $B \subset S$ be path connected subsets containing the basepoints.

Def An immersion $\iota : A \rightsquigarrow B$ is a continuous map so that $\iota(o_R) = o_S$ which acts by translation in local coordinates.
Def An embedding $e : A \hookrightarrow B$ is an injective immersion.
Def An isomorphism $f : A \to B$ is an embedding that is also a homeomorphism.

Sets that need topologies:

$\mathcal{M} = \{\text{Pointed translation structures } (S, o_S)\} / \text{isomorphism}$

$\mathcal{E}$ = Canonical translation surface "bundle" over $\mathcal{M}$
$= \{(S, o_S, s) : o_S \in S \text{ basept.}, s \in S\} / \text{isomorphism}.$

$\pi : \mathcal{E} \to \mathcal{M}$ forgetful map. Each $[S] \in \mathcal{M}$ has a canonical representative, $S = \pi^{-1}([S])$.

$\tilde{\mathcal{M}} \subset \mathcal{M}$ and $\tilde{\mathcal{E}} \subset \mathcal{E}$ structures on the disk.

Points in $\mathcal{M}$ and $\mathcal{E}$ are associated to subsets of $\tilde{\mathcal{E}}$:

$\mathcal{M} \ni (S, o_S) \sim \{(\tilde{S}, \tilde{o}_S, \tilde{s}) \in \tilde{\mathcal{E}} : \tilde{s} \in \tilde{S} \text{ is a lift of } o_S \in S\}.$
$\mathcal{E} \ni (S, o_S, s) \sim \{(\tilde{S}, \tilde{o}_S, \tilde{s}) \in \tilde{\mathcal{E}} : \tilde{s} \text{ is a lift of } s\}.$

Topology on $\tilde{\mathcal{M}}$. A closed (resp. open) disk is a subset of a translation surface which is homeomorphic to a closed (resp. open) disk.

Let K be a closed disk and U be an open disk.

Open sets in $\tilde{\mathcal{M}}$:

$\tilde{\mathcal{M}}_{\rightsquigarrow}(K) = \{P \in \tilde{\mathcal{M}} : K \underset{imm.}{\rightsquigarrow} P\}$ $\tilde{\mathcal{M}}_{\not\hookrightarrow}(U) = \{P \in \tilde{\mathcal{M}} : U \not\hookrightarrow P\}$
$\tilde{\mathcal{M}}_{\hookrightarrow}(K) = \{P \in \tilde{\mathcal{M}} : K \underset{emb.}{\hookrightarrow} P\}$ $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(U) = \{P \in \tilde{\mathcal{M}} : U \not\rightsquigarrow P\}$

Topology on $\tilde{\mathcal{E}}$: The coarsest topology so that $\pi : \tilde{\mathcal{E}} \to \tilde{\mathcal{M}}$ is continuous and so that $\forall$ closed disks K containing a non-empty open set $U \subset K^\circ$ we have

$\mathcal{E}_{\rightsquigarrow}(K, U) = \{(P, p) \in \tilde{\mathcal{E}} : \exists \iota : K \rightsquigarrow P \text{ and } p \in \iota(U)\}$

Theorem. The topologies on $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{E}}$ are locally compact, second countable, and metrizable.

Topologies on $\mathcal{M}$ and $\mathcal{E}$ are obtained through the identification with closed subsets of $\tilde{\mathcal{E}}$, which we topologize using the Chabauty-Fell topology.
$\underset{w/work}{\Longrightarrow}$ $\mathcal{M}$ and $\mathcal{E}$ are locall-compact, metrizable.

III. Examples illustrating using the topology

Philosophy: In studying the geometry and dynamics of translation surface, we need to be able to push geometric objects between a surface and its approximates.

Proposition (Stability of cylinders)
If S has a cylinder and $S_n \to S$ as $n \to \infty$ then S_n has a cylinder for n large.

Thm (Infinite variant of Masur's Criterion)
(Follows already from work of Treviño)

Let S be a translation surface of area 1 and infinite topological type. Suppose $\exists$ sequences $o_n \in S$ and $t_n \uparrow \infty$ so that $g_{t_n}(S, o_n)$ converge in $\mathcal{M}$ to an area 1 surface (S_∞, o_∞). Then the vertical flow on S is ergodic.

IV. Fundamental results about the topologies.

Thm (Basepoint change is continuous)
The map $\mathcal{E} \to \mathcal{M}$; $(S, o_S, s) \mapsto (S, s)$ is continuous. (Related maps also continuous.)
(new basepoint)

Thm (Joint continuity of immersions)
Let U be an open disk in a translation surface.
Let $\mathcal{I}(U) = \{S \in \mathcal{M} : U \rightsquigarrow S\}$. Then the map
$\mathcal{I}(U) \times U \to \mathcal{E}$; $((S, o_S), u) \mapsto (S, o_S, \iota(u))$
where $\iota : U \rightsquigarrow S$ is continuous.

Thm The $GL(2, \mathbb{R})$ actions on $\mathcal{M}$ and $\mathcal{E}$ are continuous.

Cor If $S_n \in \mathcal{M}$ is a sequence converging to S, each S_n admits an affine automorphism φ_n with derivatives $A_n \in GL(2, \mathbb{R})$ converging to A, and $(S_n, o_n, \varphi_n(o_n))$ converges in $\mathcal{E}$, then S admits a affine automorphism φ with $D\varphi = A$ so that φ carries the basepoint to the limit.

Thm For all $\varepsilon > 0$, the set of surfaces $S \in \mathcal{M}$ so that the injectivity radius at the basepoint is $\geq \varepsilon$ is compact.

Lazy limits:

Def A simply connected (pointed) translation surface $\tilde{S} \in \tilde{\mathcal{M}}$ is maximal if $\tilde{S} \hookrightarrow \tilde{S}'$ implies $\tilde{S} = \tilde{S}'$.

Def A subset $R \subset \mathbb{R}^2$ is a basic region if
① R° is a non-empty convex set.
② $R \cap \partial R$ is a countable union of pairwise disjoint open intervals in lines called edges.

Def (Lazy convergence of basic regions)
Let R^n be a sequence of basic regions and R be another. A lazy identification is a choice for each edge $e \subset R$ of an $N = N(e)$ and edges e^n for $n > N$. We say $R_n \to R$ lazily if
ⓐ $\overline{R_n} \to \overline{R}$ in the Chabauty-Fell topology
ⓑ $\forall$ edges e of R, $\overline{e^n} \to \overline{e}$ in C-F.

Def An assembly datum is a triple (G, R, E) where.
i) G is a connected multigraph with vertex set $\mathcal{V}(G)$, edge set $\mathcal{E}(G)$ and base vertex $v_0(G) \in \mathcal{V}(G)$.
ii) $\forall v \in \mathcal{V}(G)$, $R(v)$ is a region, and $\vec{vw} \mapsto E(\vec{vw})$ is a bijection from $\mathrm{Link}(v)$ to edges of $R(v)$.
iii) $\forall \overline{vw}$, $E(\vec{vw})$ and $E(\overleftarrow{vw})$ differ by translation.
iv) $\vec{0} \in R(v_0(G))^\circ$.

Def (Lazy Convergence) Let (G^n, R^n, E^n) be a sequence of gluing data and (G, R, E) be another. They converge lazily if:
1) $\forall v \in \mathcal{V}(G)$ $\exists N = N(v)$ and a choice of $v^n \in \mathcal{V}(G^n)$ for $n > N$.
2) $\forall \vec{e} = \vec{vw}$ $\exists N = N(\vec{e}) \geq N(v)$ and a choice of $\vec{e}^n \in \mathrm{Link}(v^n)$ determining a lazy convergence of $R(v^n)$ to $R(v)$.
3) We have $v_0(G)^n = v_0(G^n)$.

Thm Let S_n be assembled from (G^n, R^n, E^n) which converges lazily to (G, R, E), assembling to S. If $\tilde{S}$ is maximal, then $S_n \to S$.