Immersions, Embeddings, Isomorphisms

Let (R, o_R) and (S, o_S) be pointed translation surfaces and let $A \subset R$ and $B \subset S$ be path connected subsets containing the basepoints.

<u>Def</u> An <u>immersion</u> $\iota : A \rightsquigarrow B$ is a continuous map so that $\iota(o_R) = o_S$ which acts by translation in local coordinates

<u>Def</u> An <u>embedding</u> $e : A \hookrightarrow B$ is an injective immersion.

<u>Def</u> An <u>isomorphism</u> $f : A \to B$ is an embedding that is also a homeomorphism.

Sets that need topologies:

$$\mathcal{M} = \{\text{Pointed translation structures } (S, o_S)\} / \text{isomorphism.}$$

$$\mathcal{E} = \text{Canonical translation surface ``bundle'' over } \mathcal{M}$$
$$= \{(S, o_S, s) : o_S \in S \text{ basept.}, s \in S\} / \text{isomorphism.}$$

$\pi : \mathcal{E} \to \mathcal{M}$ forgetful map. Each $[S] \in \mathcal{M}$ has a canonical representative, $S = \pi^{-1}([S])$.

$\tilde{\mathcal{M}} \subset \mathcal{M}$ and $\tilde{\mathcal{E}} \subset \mathcal{E}$ structures on the disk.

Points in $\mathcal{M}$ and $\mathcal{E}$ are associated to subsets of $\tilde{\mathcal{E}}$:

$$\mathcal{M} \ni (S, o_S) \sim \{(\tilde{S}, \tilde{o}_S, \tilde{s}) \in \tilde{\mathcal{E}} : \tilde{s} \in \tilde{S} \text{ is a lift of } o_S \in S\}.$$
$$\mathcal{E} \ni (S, o_S, s) \sim \{(\tilde{S}, \tilde{o}_S, \tilde{s}) \in \tilde{\mathcal{E}} : \tilde{s} \text{ is a lift of } s\}.$$

Topology on $\widetilde{\mathcal{M}}$. A closed (resp. open) disk is a subset of a translation surface which is homeomorphic to a closed (resp. open) disk.

Let K be a closed disk and U be an open disk.

Open sets in $\widetilde{\mathcal{M}}$:

$$\widetilde{\mathcal{M}}_{\rightsquigarrow}(K)=\{P\in\widetilde{\mathcal{M}} : K \underset{\text{imm.}}{\rightsquigarrow} P\} \qquad \widetilde{\mathcal{M}}_{\not\hookrightarrow}(U)=\{P\in\widetilde{\mathcal{M}} : U \not\hookrightarrow P\}$$

$$\widetilde{\mathcal{M}}_{\hookrightarrow}(K)=\{P\in\widetilde{\mathcal{M}} : K \underset{\text{emb.}}{\hookrightarrow} P\} \qquad \widetilde{\mathcal{M}}_{\not\rightsquigarrow}(U)=\{P\in\widetilde{\mathcal{M}} : U \not\rightsquigarrow P\}$$

Topology on $\widetilde{\mathcal{E}}$: The coarsest topology so that $\pi : \widetilde{\mathcal{E}} \to \widetilde{\mathcal{M}}$ is continuous and so that $\forall$ closed disks K containing a non-empty open set $U \subset K^\circ$ we have

$$\mathcal{E}_{\rightsquigarrow}(K,U)=\{(P,p)\in\widetilde{\mathcal{E}} : \exists\iota : K\rightsquigarrow P \text{ and } p\in\iota(U)\}$$

Theorem. The topologies on $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{E}}$ are locally compact, second countable, and metrizable.

Topologies on $\mathcal{M}$ and $\mathcal{E}$ are obtained through the identification with closed subsets of $\widetilde{\mathcal{E}}$, which we topologize using the Chabauty-Fell topology.

$\underset{\text{with work}}{\Longrightarrow}$ $\mathcal{M}$ and $\mathcal{E}$ are locally-compact, metrizable.