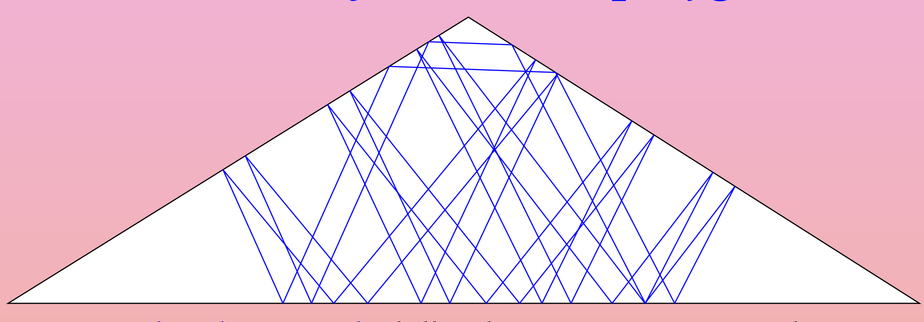
29th Summer Conference on
TOPOLOGY and its APPLICATIONS
July 23-26, 2014
College of Staten Island, CUNY

Pat Hooper
(City College of NY and CUNY Grad Center)

Detecting vanishing periodic billiard trajectories in polygons



A closed or periodic billiard trajectory in a triangle.

Open Question: Does every polygon admit a periodic billiard trajectory?

Theorem (Schwartz): Every triangle with largest angle less than 100° has a periodic billiard trajectory.

A phenomenon discovered by Schwartz:

Let P be a polygon in the plane. We define $L(P)$ to be the length of the shortest periodic billiard trajectory in P or ∞ if none exists.

Theorem (Schwartz): The function L is not locally finite at the $(30, 60, 90)$ triangle.

Parallels for Isosceles triangles

Theorem (H-Schwartz): There is a neighborhood of the set of isosceles triangles so that every triangle in the neighborhood has a periodic billiard trajectory.

Conjecture (H-Schwartz): The function L is not locally finite at the triangle with angles

$$\left(\frac{\pi}{2^k}, \frac{\pi}{2^k}, \frac{\pi(2^{k-1}-1)}{2^{k-1}}\right)$$

for any integer $k \geq 3$.

Todays Goal:

I'll give some ideas that I expect will lead to a proof of this conjecture.

Idea 1: Consider real analytic paths of polygons.

Theorem (Criterion for non-local boundedness):
Let $t \mapsto P_t$ be real analytic and defined on $[0, a)$ for some $a > 0$. If L is locally bounded at P_0, then there is an $\epsilon > 0$ and a single orbit type $\mathcal{O}$ so that P_t lies in the orbit tile of $\mathcal{O}$ for $0 < t < \epsilon$.

A related classification problem:
Given $t \mapsto P_t$ real analytic, find all $\mathcal{O}$ so that there is an $\epsilon > 0$ so that P_t lies in the orbit tile when $0 < t < \epsilon$.

Idea 2: A translation surface is a topological surface with an atlas of charts to the plane so that the transition functions are translations.

Translation surfaces can sometimes be used to answers to the following questions:

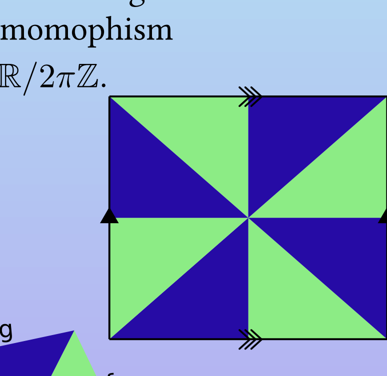
1. Classify the orbit types of periodic billiard trajectories in a polygon P_0.

For a real analytic path $t \mapsto P_t$,

2. Classify the orbit types so that P_t lies in the orbit tile for $0 \leq t < \epsilon$ for some $\epsilon > 0$.
3. Classify the orbit types so that P_t lies in the orbit tile for $0 < t < \epsilon$ for some $\epsilon > 0$.

The translation surface associated to a polygon:

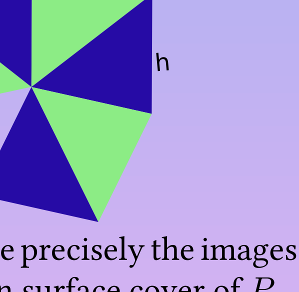
Given a polygon P, let DP denote the double of P across its boundary.



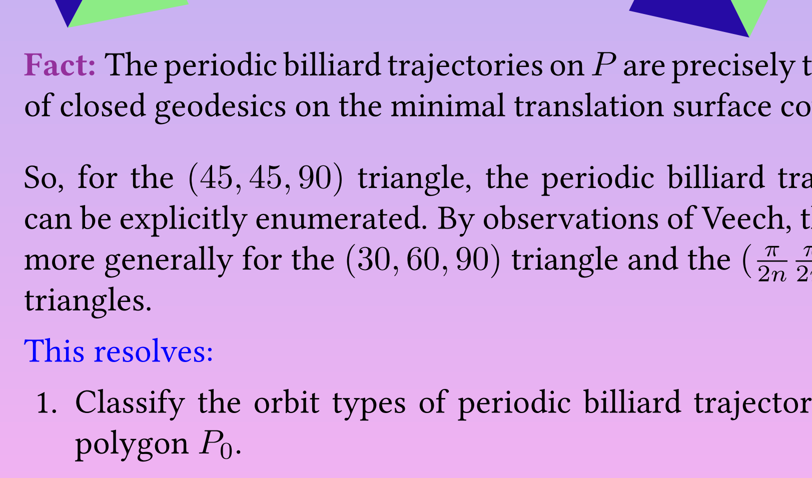
- DP is a Eucidean cone surface.
- There is a folding map $DP \to P$.
- A periodic billiard trajectory in P lifts to a closed geodesic in DP.
- Closed geodesics on DP have trivial rotational holonomy.

Let DP^* denote the double of P with its cone singularities removed. Rotational holonomy gives a group homomophism

$$hol : \pi_1(DP^*) \to \mathbb{R}/2\pi\mathbb{Z}.$$



The cover of DP^* associated to $ker(hol)$ is the minimal translation surface cover.



Fact: The periodic billiard trajectories on P are precisely the images of closed geodesics on the minimal translation surface cover of P.

So, for the $(45, 45, 90)$ triangle, the periodic billiard trajectories can be explicitly enumerated. By observations of Veech, this holds more generally for the $(30, 60, 90)$ triangle and the $(\frac{\pi}{2n} \frac{\pi}{2n}, *)$ triangles.

This resolves:

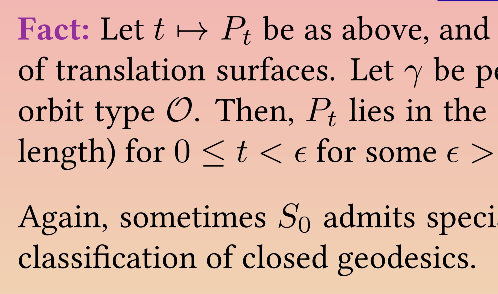
1. Classify the orbit types of periodic billiard trajectories in a polygon P_0.

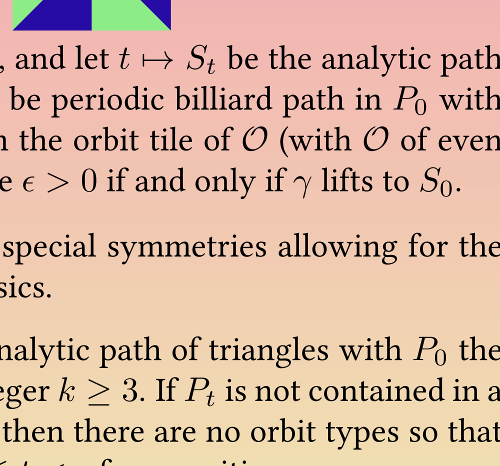
Translation surfaces associated to a real analytic path of polygons:

Let $t \mapsto P_t$ be a real analytic path of polygons defined on $[0, a)$ for some $a > 0$. Then, for each t, we get a rotational holonomy homomorphism:

$$hol_t : \pi_1(DP_t^*) \to \mathbb{R}/2\pi\mathbb{Z}.$$

The associated analytic path of translation surfaces is the family of covers of DP_t^* associated to $\bigcap_t ker(hol_t)$.





Fact: Let $t \mapsto P_t$ be as above, and let $t \mapsto S_t$ be the analytic path of translation surfaces. Let γ be periodic billiard path in P_0 with orbit type $\mathcal{O}$. Then, P_t lies in the orbit tile of $\mathcal{O}$ (with $\mathcal{O}$ of even length) for $0 \leq t < \epsilon$ for some $\epsilon > 0$ if and only if γ lifts to S_0.

Again, sometimes S_0 admits special symmetries allowing for the classification of closed geodesics.

Theorem: Let P_t be a real analytic path of triangles with P_0 the $(\frac{\pi}{2^k}, \frac{\pi}{2^k}, *)$ triangle for an integer $k \geq 3$. If P_t is not contained in a line in the space of triangles, then there are no orbit types so that P_t lies in the orbit tile for $0 \leq t < \epsilon$ for a positive ϵ.

Streched limits of analytic paths of translation surfaces:

Observations:

- Translation surfaces arising from polygonal billiard tables are special: All singularities are of cone type, and the distance between singularities is bounded from below by some $\eta > 0$. We call such surfaces η-forthright.
- There is a natural definition of an real analytic path in the space of pointed η-forthright surfaces having to do with the motion of cone singularities. This singularites move around in a real analytic way, except they may tend to ∞.
- There is a naural way to identify homotopy classes within an analytic family. Closed geodesics lie in a cylinder whose circumferences and widths vary real analytically in t.
- There is a $GL(2, \mathbb{R})$ action on translation surfaces, and it preserves the forthright surfaces.
- The η-forthright surfaces form a closed set in a natural topology on the space of all pointed translation surfaces. This topology is described in a pair of articles on the ArXiv, *Immersions and translation structures on the disk* and *Immersions and the space of all translation structures*.

Theorem (Stretched limits):
Let $t \mapsto S_t$ be a real analytic path in the space of η-forthright surface defined for $t \in [0, a)$. Define the family of matrices:

$$A_t = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{t} \end{bmatrix}.$$

Then, there is a limit $S_0' = \lim_{t \to 0} A_t(S_t)$. Furthermore, the path defined by S_0' and $S_t' = A_t(S_t)$ for $t \in [0, a)$ is a real analytic path of η-forthright translation surfaces.

Theorem (Using streched limits):
Suppose a family of cylinders in S_t is assymptotically horizontal as $t \to 0$ and has width function $w(t) = ct + O(t^2)$. If the homotopy class has a limit in $S_0' = \lim_{t \to 0} A_t(S_t)$, then there is a cylinder in this homotopy class.

Remarks:

- The cylinders can be arranged to be assymptotically horizontal by rotating S_t uniformly.
- The homotopy classes can be made to converge by choosing basepoints appropriately.
- Cylinders whose width functions are $w(t) = ct^k + O(t^{k+1})$ for $k > 1$ can be detected by successively rotating and stretching k times.

Fact (Classification):
In some cases, the successive strecthed limits of an analytic family $t \mapsto S_t$ can be classified.

Example: