Renormalization in piecewise isometries

MathFest 2014 @ CUNY Graduate Center

Pat Hooper
City College of New York and CUNY Graduate Center

Dynamical systems and renormalization

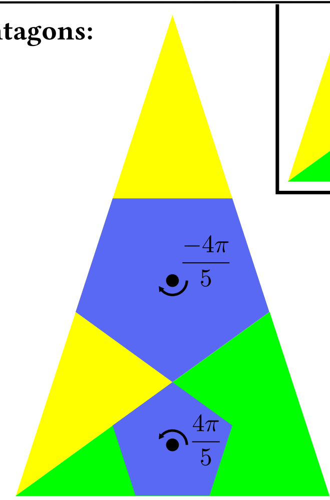
A *dynamical system* is a space together with a time independent update rule.

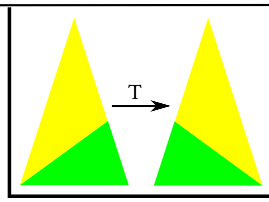
So, a map $T : X \to X$ is a dynamical system.

Renormalization is an approach to understanding certain dynamical systems. It is used to study:

- Complex dynamics (e.g., iteration of polynomials- Julia sets, Mandlebrot set)
- Flows on symmetric spaces
- ★ **Piecewise isometries**

Fixed Pentagons:





Return maps:

Let $T{:}X \to X$ be a map.

The **forward orbit** of $x \in X$ is the sequence:

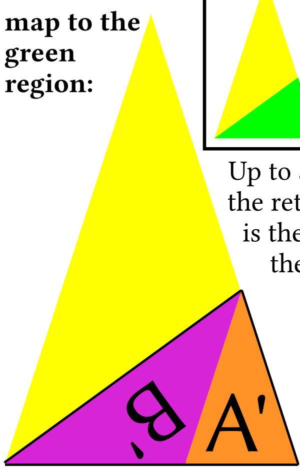
$$\{T(x),\ T^2(x) = T \circ T(x),\ T^3(x) = T \circ T \circ T(x),\ \ldots\}.$$

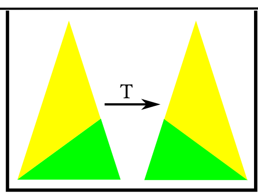
Let A be a subset of X. The **first return** of $a \in A$ to A is the first point in the forward orbit of a which lies in A.

Let $A' \subset A$ be the set of points with a first return to A.

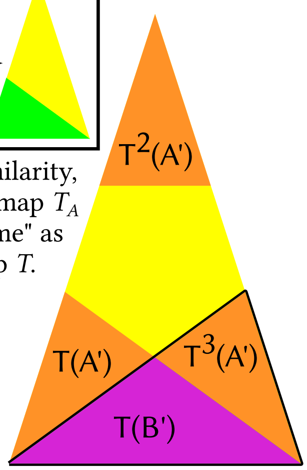
The **return map** to A is the map $T_A{:}A' \to A$ which sends a point $a \in A'$ to its first return $T_A(a)$.

The return map to the green region:





Up to a similarity, the return map T_A is the "same" as the map T.



Triangles with the same return map up to a similarity:

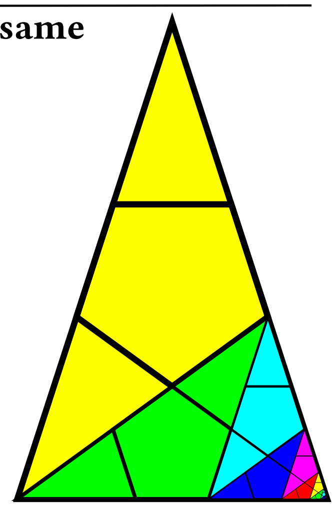
$\triangle A_0A_1B$

$\triangle A_1A_2B$

$\triangle A_2A_3B$

...

Each of these gives rise to a periodic pentagon.



Triangles with the same return map up to a similarity:

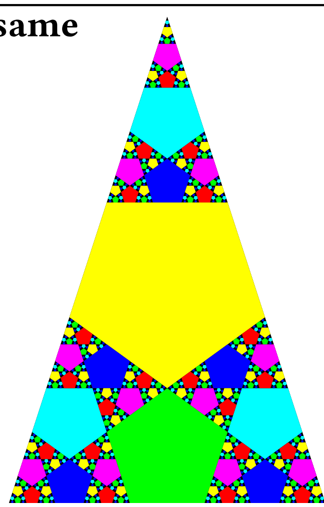
$\triangle A_0A_1B$

$\triangle A_1A_2B$

$\triangle A_2A_3B$

...

Each of these gives rise to a periodic pentagon's orbit.



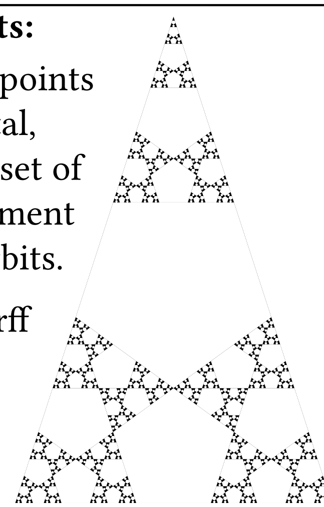
The aperiodic points:

The set of aperiodic points is a self-similar fractal, and is (roughly) the set of points in the compliment of the pentagonal orbits.

This set has Hausdorff dimension

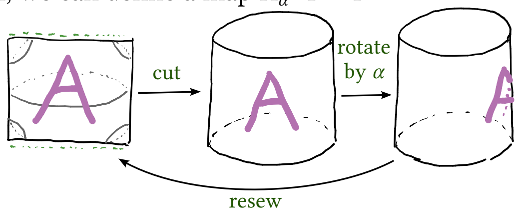
$$d = \frac{\log 2}{\log \phi} \approx 1.44,$$

where $\phi = \frac{1+\sqrt{5}}{2}$.

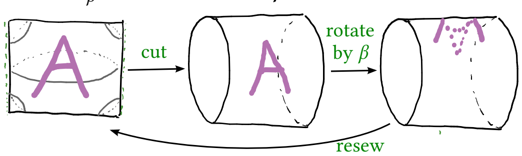


A family of piecewise isometries on the pillowcase:

Then, we can define a map $H_\alpha : P \longrightarrow P$:



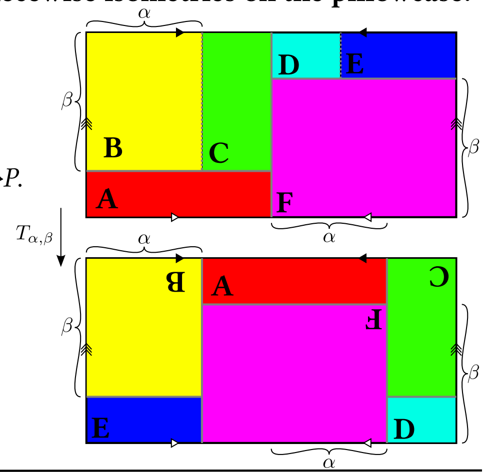
We can do the same in the vertical direction.
We define $V_\beta : P \longrightarrow P$, with $0<\beta<½$.



A family of piecewise isometries on the pillowcase:

Let $0<\alpha<½$ and $0<\beta<½$.

We define $T_{\alpha,\beta}=H_\alpha \circ V_\beta : P \longrightarrow P$.



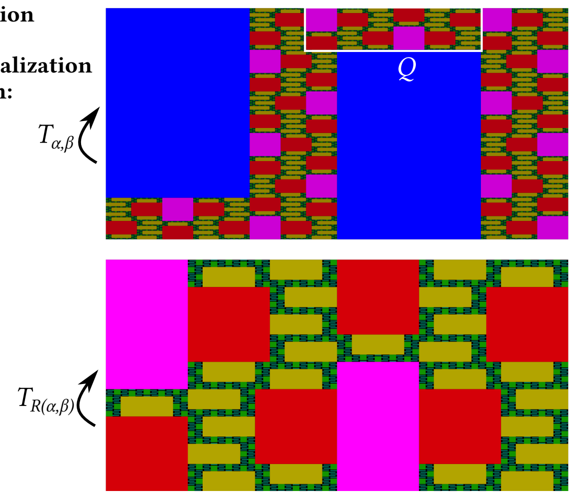
A renormalization theorem:

For $x \in \mathbb{R}$, let $nint(x)$ denote the nearest integer.
For $0<\alpha<½$ and $0<\beta<½$ irrational, define:

$$R(\alpha,\beta)=\Big(\big|\frac{\alpha}{1-2\alpha}-nint(\frac{\alpha}{1-2\alpha})\big|,\big|\frac{\beta}{1-2\beta}-nint(\frac{\beta}{1-2\beta})\big|\Big).$$

Theorem. Let α and β be irrationals satisfying $0<\alpha<½$ and $0<\beta<½$. Then, there is a rectangle Q in the pillowcase P so that the return map of $T_{\alpha,\beta}$ to Q is the same as $T_{R(\alpha,\beta)}$ up to an affine coordinate change and sewing up Q to make a pillowcase.

Illustration of the Renormalization Theorem:



Philosophy of Renormalization:

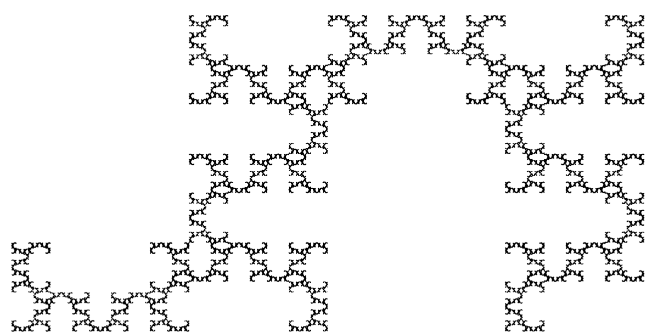
Corollary. Let α and β be irrationals satisfying $0<\alpha<½$ and $0<\beta<½$. Consider the forward R-orbit of (α, β):

$$\{ R(\alpha, \beta), R^2(\alpha, \beta)=R\circ R(\alpha, \beta), \ldots \}.$$

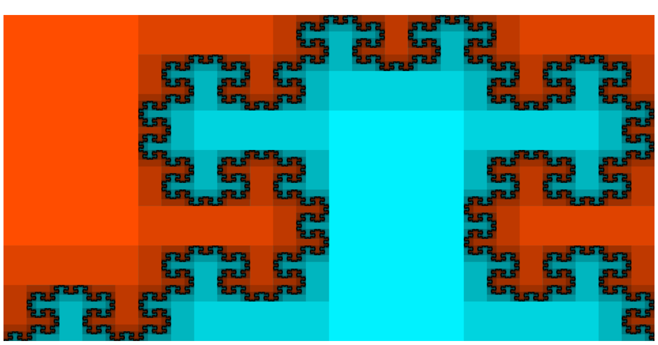
For every integer $n>0$, there is a rectangle Q_n so that the return map of $T_{\alpha,\beta}$ to Q_n is affinely conjugate to $T_{R^n(\alpha,\beta)}$.

Philosophy. The dynamical behavior of $T_{\alpha,\beta}$ is related to the dynamics of the forward R-orbit of (α, β).

If (α, β) is periodic under R, then $T_{\alpha,\beta}$ is self-similar.



If the orbit of (α, β) avoids a certain collection of rectangles in the (α, β)-plane, then the aperiodic points of $T_{\alpha,\beta}$ form a curve.



Let $(\alpha_n, \beta_n)=R^n(\alpha, \beta)$.
If $\limsup \min(\alpha_n, \beta_n)>0$, then the aperiodic points have zero area.
But there are examples with positive area: