Ergodic Theory and Statistical Mechanics
Princeton University
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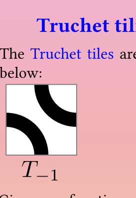
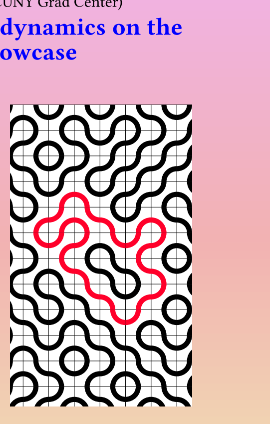
Piecewise isometric dynamics on the square pillowcase

Truchet tilings

The Truchet tiles are the two tiles below:





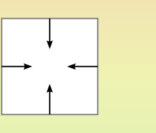
Given any function $\tau: \mathbb{Z}^2 \to \{-1, 1\}$, we can construct a Truchet tiling by placing the tile $T_{\tau(m,n)}$ centered at (m,n) for every $m, n \in \mathbb{Z}$.

General Question: Pick an edge e in the tiling of the plane by squares. Given a translation invariant probability measure on the space of Truchet tilings, what is the probability that the curve through e in a randomly chosen tiling closes up?

Dynamics and tilings

Let T be the space of all Truchet tilings, i.e., $\tau: \mathbb{Z}^2 \to \{\pm 1\}$. We define

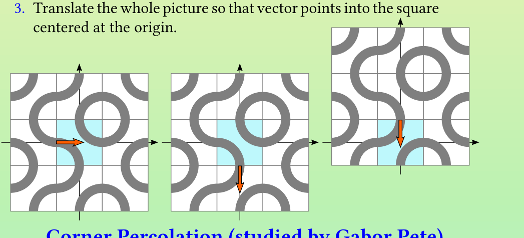
$$N = \{\text{up}, \text{down}, \text{left}, \text{right}\}.$$



We think of these as being four inward pointing vectors at the unit square centered at the origin.

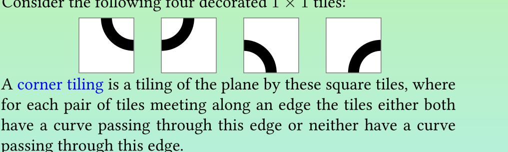
The update map $f: T \times N \to T \times N$ is given by:

1. We start with a tiling and a tangent to a curve pointed into the square centered at the origin.
2. Slide the vector along the curve for a quarter turn.
3. Translate the whole picture so that the vector points into the square centered at the origin.

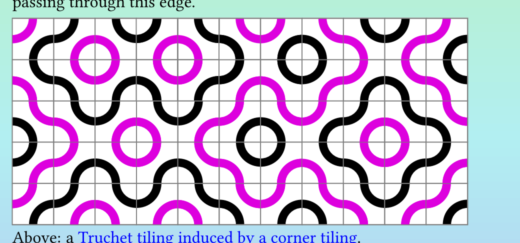


Corner Percolation (studied by Gabor Pete)

Consider the following four decorated 1×1 tiles:



A corner tiling is a tiling of the plane by these square tiles, where for each pair of tiles meeting along an edge the tiles either both have a curve passing through this edge or neither have a curve passing through this edge.



Above: A Truchet tiling induced by a corner tiling.

Tilings from shift spaces

Let Ω denote the full shift space on the alphabet $\{\pm 1\}$.

Proposition. The Truchet tiling determined by $\tau: \mathbb{Z}^2 \to \{\pm 1\}$ is induced by a corner tiling if and only if there are $\omega, \eta \in \Omega$ so that

$$\tau(m,n) = \omega_m \eta_n \quad \text{for all } m, n \in \mathbb{Z}.$$

The map $(\omega, \eta) \mapsto \tau$ is two-to-one.

Furthermore, the action $f: T \times N \to T \times N$ preserves the space of Truchet tilings induced by corner tilings and lifts to a map

$$\tilde{f}: \Omega^2 \times N \to \Omega^2 \times N.$$

Maximal entropy corner tilings

Theorem (Gabor Pete). Select $\omega, \eta \in \Omega$ with each ω_m and η_n chosen from $\{\pm 1\}$ according to a fair coin flip. Then, with probability one, every curve of the tiling defined by the function $\tau(m,n) = \omega_m \eta_n$ is closed.

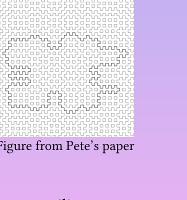
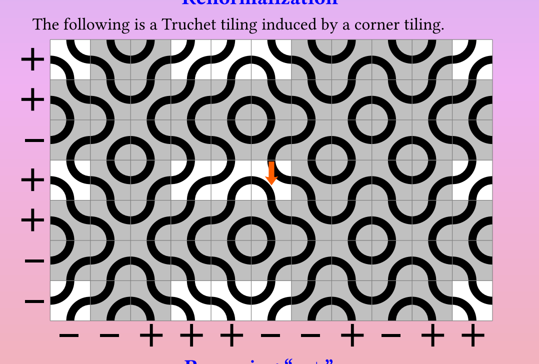


Figure from Pete's paper

Renormalization

The following is a Truchet tiling induced by a corner tiling.



Removing "−+"

Our renormalization procedure involves removing all words of the form −+ from elements of Ω.

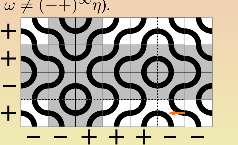
We define the collapsing map c from a subset $C \subset \Omega$ to Ω.

This operation only works if certain conditions on ω are satisfied. We define $C \subset \Omega$ to be those $\omega \in \Omega$ so that

1. The words $\omega_{2n}\omega_{2n+1}$ are not "−+".
2. The bi-infinite word ω is not eventually alternating in either direction.

The return map

The set of renormalizable states are $B = C \times C \times N \subset \Omega^2 \times N$.



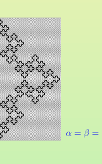
Let $\tilde{f}_B: B \to B$ be the first return map.

We define $\rho: B \to \Omega^2 \times N$ to be $\rho(\omega, \eta, \mathbf{v}) = (c(\omega), c(\eta), \mathbf{v})$.

We think of the return map $\tilde{f}_B$ and ρ as defining a renormalization of $\tilde{f}$.

Tilings determined by rotations

For irrationals $\alpha, \beta \in (0, \frac{1}{2})$. For $x, y \in [0,1)$ define



$$\omega^x_m = \begin{cases} 1 & \text{if } x + m\alpha \pmod 1 \text{ lies in } [0, \frac{1}{2}) \\ -1 & \text{if } x + m\alpha \pmod 1 \text{ lies in } [\frac{1}{2}, 1). \end{cases}$$

$$\eta^y_n = \begin{cases} 1 & \text{if } y + n\beta \pmod 1 \text{ lies in } [0, \frac{1}{2}) \\ -1 & \text{if } y + n\beta \pmod 1 \text{ lies in } [\frac{1}{2}, 1). \end{cases}$$

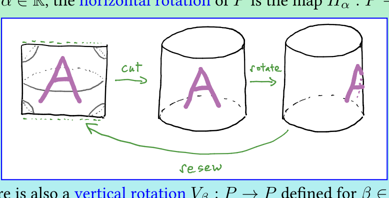
Proposition. The map $(x, y, \mathbf{v}) \mapsto (\omega^x_m, \eta^y_n, \mathbf{v})$ conjugates the action of a piecewise translation $\widetilde{T}_{\alpha,\beta}$ on $\mathbb{T}^2 \times N$ (four copies of the square torus) to the action of $\tilde{f}$ on $\Omega^2 \times N$.

There is a group of order 4 of orientation preserving isometries of $\mathbb{T}^2 \times N$ which commutes with the action of $\widetilde{T}_{\alpha,\beta}$ for all α and β.

The quotient action

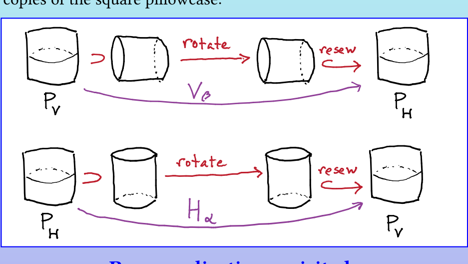
The square pillowcase, P, is the double of the square $[0, \frac{1}{2}]^2$ across its boundary.

For $\alpha \in \mathbb{R}$, the horizontal rotation of P is the map $H_\alpha: P \to P$:



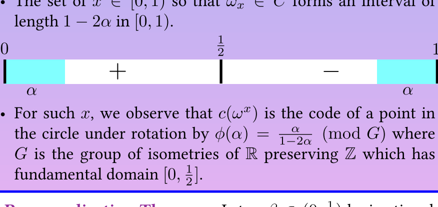
There is also a vertical rotation $V_\beta: P \to P$ defined for $\beta \in \mathbb{R}$.

The quotient action $T_{\alpha,\beta}$ is defined on the union $P_h \cup P_v$ of two copies of the square pillowcase:



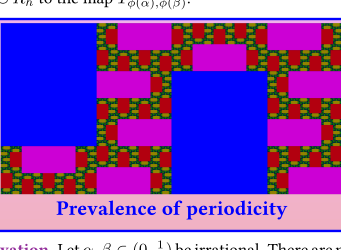
Renormalization revisited

- The map $\tilde{f}: \Omega \times \Omega \times N \to \Omega \times \Omega \times N$ was renormalized by $\rho(\omega, \eta, \mathbf{v}) = (c(\omega), c(\eta), \mathbf{v})$, where $c: C \to \Omega$ was the collapsing map.
- Fix $\alpha \in (0, \frac{1}{2})$ irrational. Each $x \in [0,1)$ determines an ω^x in Ω via coding orbits under rotation by α.
- The set of $x \in [0,1)$ so that $\omega_x \in C$ forms an interval of length $1 - 2\alpha$ in $[0,1)$.



- For such x, we observe that $c(\omega^x)$ is the code of a point in the circle under rotation by $\phi(\alpha) = \frac{\alpha}{1-2\alpha} \pmod G$ where G is the group of isometries of $\mathbb{R}$ preserving $\mathbb{Z}$ which has fundamental domain $[0, \frac{1}{2}]$.

Renormalization Theorem. Let $\alpha, \beta \in (0, \frac{1}{2})$ be irrational. Consider $T_{\alpha,\beta}: P_v \cup P_h \to P_v \cup P_h$. There is a pair of open rectangles $R_v \subset P_v$ and $R_h \subset P_h$ and an affine inclusion $\psi: R_v \cup R_h \to P_v \cup P_h$ which conjugates the first return map to $R_v \cup R_h$ to the map $T_{\phi(\alpha),\phi(\beta)}$.



Prevalence of periodicity

Observation. Let $\alpha, \beta \in (0, \frac{1}{2})$ be irrational. There are periodic orbits of $T_{\alpha,\beta}$ arbitrary high period. Periodic rectangles are dense in the domain.

Theorem. For a.e. pair of irrationals (α, β), the periodic points have full measure in the domain.

Theorem. For any $\epsilon > 0$, there are irrationals α and β so that the measure of the periodic points is less that ϵ.

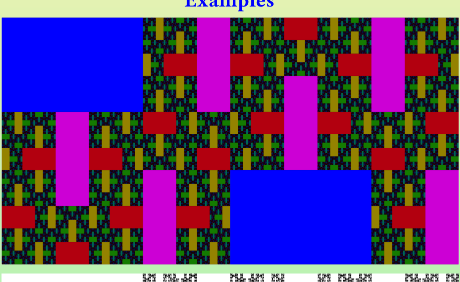
What structure allows for the proof of the theorems?

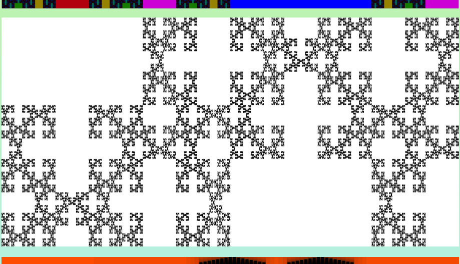
- Fix α and β irrational. From renormalization, we get a pair of nested sequences of rectangles:

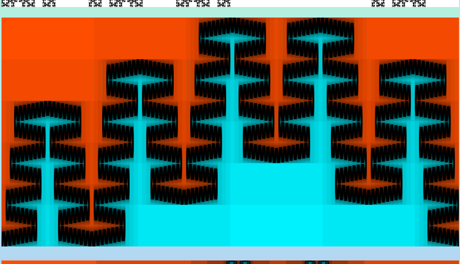
$$\{R^n_v : n \in \mathbb{N}\} \quad \text{and} \quad \{R^n_h : n \in \mathbb{N}\}.$$

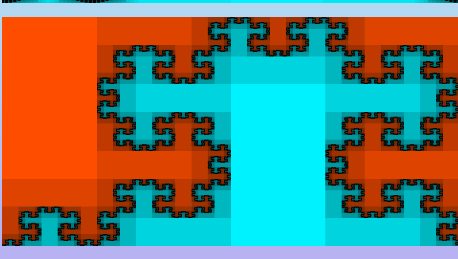
- The rectangles distinguish periodicity from aperiodicity. A non-singular trajectory is periodic if and only if it does not intersect some $R^n_v \cup R^n_h$.
- The return times to $R^n_v \cup R^n_h$ are piecewise constant at most eight possible return times.
- The vector of return times is governed by an $SL(4, \mathbb{Z})$ valued cocycle over the renormalization dynamics, $\phi \times \phi$. There is a limiting formula for the measure for the aperiodic points in terms of this cocycle.

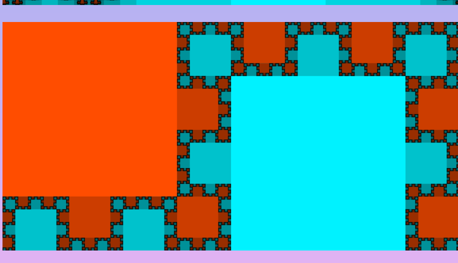
Examples

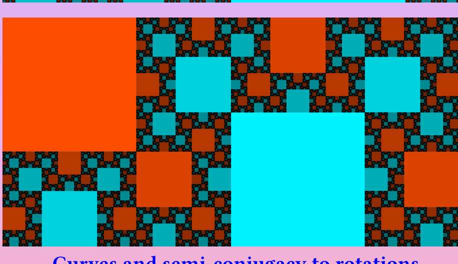












Curves and semi-conjugacy to rotations

Let $\alpha \in (0, \frac{1}{2}) \setminus \mathbb{Q}$. Recall $\phi(\alpha) = \frac{\alpha}{1-2\alpha} \pmod G$.

Observe there exist a unique integer $n = n(\alpha) \geq 0$ and a sign $s = s(\alpha) \in \{\pm 1\}$ so that

$$\phi(\alpha) = s(\frac{\alpha}{1-2\alpha} - n).$$

The even continued fraction sequence of α is the sequence of pairs:

$$\{(n \circ \phi^k(\alpha), s \circ \phi^k(\alpha)) : k \geq 0\}.$$

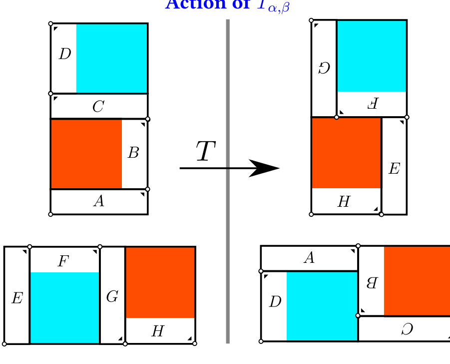
Semi-conjugacy theorem.

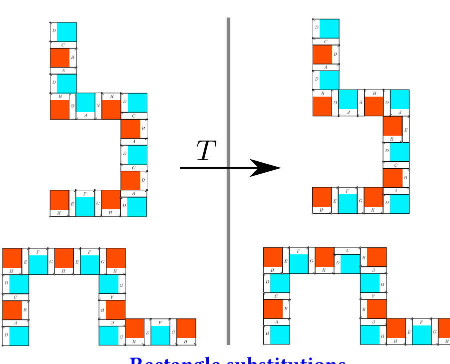
Let $\alpha, \beta \in (0, \frac{1}{2}) \setminus \mathbb{Q}$, and let $\{(m_k, r_k)\}$ and $\{(n_k, s_k)\}$ be their even continued fraction sequences.

If the sign sequences $\{r_k\}$ and $\{s_k\}$ coincide, then there is a continuous map from a pair of circles $h: \mathbb{T}_v \cup \mathbb{T}_h \to P_v \cup P_h$ and an orientation preserving isometry $R: \mathbb{T}_v \cup \mathbb{T}_h \to \mathbb{T}_v \cup \mathbb{T}_h$ so that the following hold:

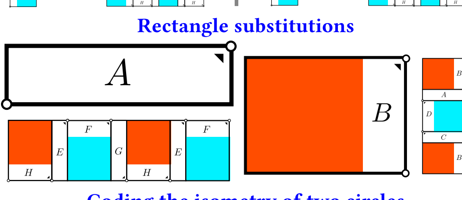
1. $h(\mathbb{T}_v \cup \mathbb{T}_h)$ is closure of the set of aperiodic points of $T_{\alpha,\beta}$.
2. $h \circ R(t) = T_{\alpha,\beta} \circ h(t)$ whenever $T_{\alpha,\beta} \circ h(t)$ is defined.
3. Whenever $T_{\alpha,\beta} \circ h(t)$ is undefined, there is a sequence $\langle t_n \rangle \to t$ with $T_{\alpha,\beta} \circ h(t_n)$ defined, and for any such sequence we have
$$h \circ R(t) = \lim_{n \to \infty} T_{\alpha,\beta} \circ h(t_n).$$
4. The even continued fraction sequence of the rotation number of R^2 is $\{(m_k + n_k, s_k)\}$.
5. If $s_k = 1$ infinitely often, then h is injective.

Action of $T_{\alpha,\beta}$

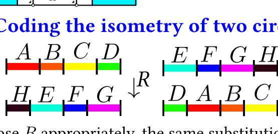




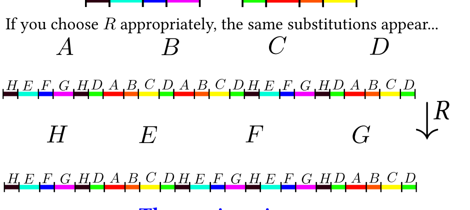
Rectangle substitutions



Coding the isometry of two circles



If you choose R appropriately, the same substitutions appear...



The semi-conjugacy