Topologizing the space of all translation surfaces

CUNY Graduate Center
Complex Analysis and Dynamics Seminar
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Based on the preprints:

- Immersions and translation structures on the disk; arXiv:1309.4795.
- Immersions and the space of all translation structures; arXiv:1310.5193.

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What is a translation surface?

A translation surface is a topological surface equipped with an atlas of charts to the plane where the transition functions are translations.

Sources of examples:

- A Riemann surface equipped with a holomorphic 1-form.
- Surfaces built by gluing together Euclidean polygons by translations.
- Polygonal billiards.
- Suspensions of interval exchange maps.

Goals for the talk:

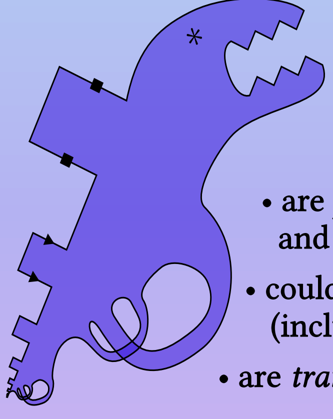
1. Place a canonical and useful topology on the space $\mathcal{M}$ of all (pointed) translation surfaces.
2. Describe some facts about the topology.
3. Describe some dynamical consequences.

Approach to topologizing:

1. Topologize $\tilde{\mathcal{M}}$, the space of (pointed) translation surfaces homeomorphic to an open disk.
2. Topologize $\tilde{\mathcal{E}}$, the disk bundle over $\tilde{\mathcal{M}}$.
3. Topologize $\mathcal{M}$, the space of all (pointed) translation surfaces, (and $\mathcal{E}$, the surface bundle over $\mathcal{M}$).

The set of **all** translation surfaces:

A translation surface is a topological surface equipped with an atlas of charts to the plane where the transition functions are translations.



Surfaces in the set of all translation surfaces $\mathcal{M}$...

- have no singularities, but are incomplete (with few exceptions).
- are *pointed* (have a basepoint $*$) and are connected.
- could have any topological type (including infinite type).
- are *translation equivalence* classes.

Two pointed translation surfaces, S_1 and S_2, are *translation equivalent (equal)* if there is a homeomorphism $h : S_1 \to S_2$ which respects the basepoints and is a translation in local coordinates.

Aside: Order Theory

Immersions place a partial order on the space $\tilde{\mathcal{M}}$ of all simply connected translation surfaces.

Theorem. The set $\tilde{\mathcal{M}} \cup \{0\}$, where 0 denotes the degenerate, single point "translation surface" is a complete lattice, i.e., each subset of $\tilde{\mathcal{M}} \cup \{0\}$ has a supremum and an infimum in $\tilde{\mathcal{M}} \cup \{0\}$.

Convergence in $\tilde{\mathcal{M}}$

Let $\tilde{S}_n$ be a sequence in $\tilde{\mathcal{M}}$. Then $\tilde{S}_n$ converges to $\tilde{S}$ if and only if both of the following hold:

1. For every closed topological disk $K \subset \tilde{S}$ containing the basepoint, $K \rightsquigarrow \tilde{S}_n$ for n sufficiently large.
2. For all $U \in \tilde{\mathcal{M}}$, if $U \rightsquigarrow \tilde{S}_n$ for infinitely many n, then $U \rightsquigarrow \tilde{S}$.

Example of convergence:

For $n \geq 1$, let $R_n \subset \mathbb{C}$ be the n-th roots of unity. Let $S_n = \mathbb{C} \setminus R_n$ with basepoint at the origin.

Then, the sequence of universal covers $\tilde{S}_n$ converges to the unit disk.

The disk bundle:

The disk bundle over $\tilde{\mathcal{M}}$ is

$$\tilde{\mathcal{E}} = \{(\tilde{S}, p) : p \in \tilde{S} \in \tilde{\mathcal{M}}\}.$$

The topology on the disk bundle:

Let $(\tilde{S}_n, p_n)$ be a sequence in $\tilde{\mathcal{E}}$. Then, the sequence converges to $(S, p) \in \tilde{\mathcal{E}}$ if

1. $\tilde{S}_n \to \tilde{S}$ in $\tilde{\mathcal{M}}$, and
2. for one (equivalently all) closed disk $K \subset \tilde{S}$ with $p \in K^\circ$, the immersions $\iota_n : K \rightsquigarrow \tilde{S}_n$ satisfy $d_n(p_n, \iota_n(p)) \to 0$.

Topologizing the space of all surfaces:

Let $S_n \in \mathcal{M}$ be a sequence of translation surfaces, and let $S \in \mathcal{M}$ be a potential limit. Let s_n and s be their basepoints and let $\tilde{S}_n$ and $\tilde{S}$ be their universal covers.

Then, S_n *converges* to S in $\mathcal{M}$ if

A. $\tilde{S}_n$ converges to $\tilde{S}$ in $\tilde{\mathcal{M}}$, and

B. a point $\tilde{p} \in \tilde{S}$ is a lift of $s \in S$ if and only if there is a sequence $\tilde{p}_n \in \tilde{S}_n$ so that $(\tilde{S}_n, \tilde{p}_n)$ converges to $(\tilde{S}, \tilde{p})$ in $\tilde{\mathcal{E}}$.

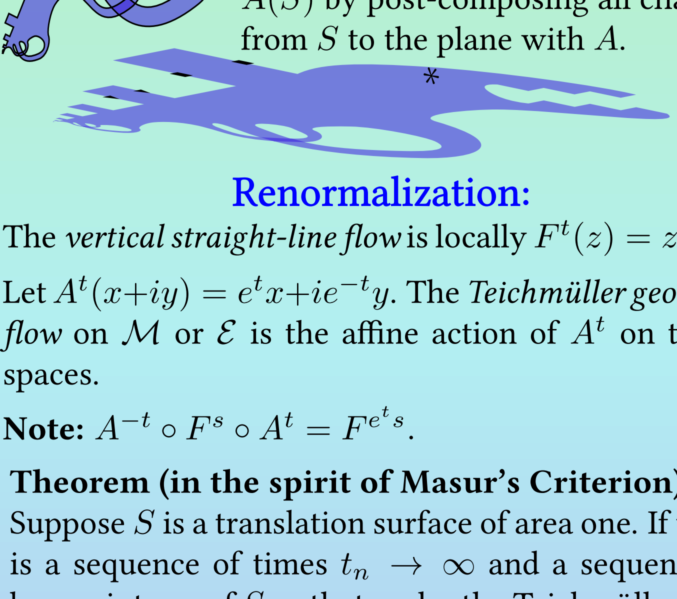
We also topologize $\mathcal{E}$, the surface bundle over $\mathcal{M}$...

The *immersive* topologies are nice:

Theorem. The topologies on $\tilde{\mathcal{M}}$, $\tilde{\mathcal{E}}$, $\mathcal{M}$, and $\mathcal{E}$ are second countable and Hausdorff.

Compactness Theorem. For any $\epsilon > 0$, the set of surfaces in $\mathcal{M}$ or $\tilde{\mathcal{M}}$ for which the basepoint has an open ϵ-neighborhood isometric to the open ϵ-ball in the plane is compact.

Dynamics:



Let S be a translation surface, and let u be a unit complex number. The *straight-line flow* is given in local coordinates by $F^t(z) = z + tu$, with $t \in \mathbb{R}$.

$GL(2,\mathbb{R})$ acts on translation surfaces. If $A \in GL(2,\mathbb{R})$, then we obtain $A(S)$ by post-composing all charts from S to the plane with A.

Renormalization:

The *vertical straight-line flow* is locally $F^t(z) = z + it$.

Let $A^t(x+iy) = e^t x + ie^{-t} y$. The *Teichmüller geodesic flow* on $\mathcal{M}$ or $\mathcal{E}$ is the affine action of A^t on these spaces.

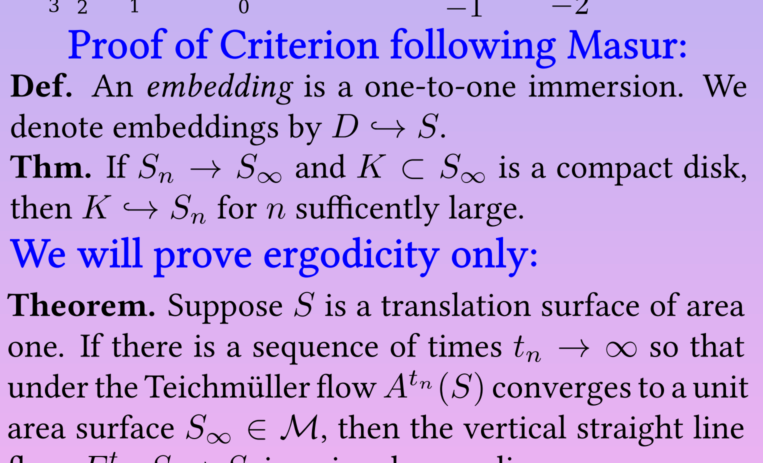
Note: $A^{-t} \circ F^s \circ A^t = F^{e^t s}$.

Theorem (in the spirit of Masur's Criterion). Suppose S is a translation surface of area one. If there is a sequence of times $t_n \to \infty$ and a sequence of basepoints s_n of S so that under the Teichmüller flow $A^{t_n}(S, s_n)$ converges to a unit area surface in $\mathcal{M}$, then the vertical straight line flow is uniquely ergodic.

There are two proofs:

- One uses more general work of Rodrigo Teviño.
- A second mirrors Masur's proof from the finite genus case.

Example surfaces:



Proof of Criterion following Masur:

Def. An *embedding* is a one-to-one immersion. We denote embeddings by $D \hookrightarrow S$.

Thm. If $S_n \to S_\infty$ and $K \subset S_\infty$ is a compact disk, then $K \hookrightarrow S_n$ for n sufficently large.

We will prove ergodicity only:

Theorem. Suppose S is a translation surface of area one. If there is a sequence of times $t_n \to \infty$ so that under the Teichmüller flow $A^{t_n}(S)$ converges to a unit area surface $S_\infty \in \mathcal{M}$, then the vertical straight line flow, $F^t : S \to S$, is uniquely ergodic.

Proof of Criterion following Masur:

The individual ergodic theorem says that for any integrable f, there is a set $B = B(f) \subset S$ of full measure so that the averages

$$\mathrm{avg}_+(x) = \int_0^T f \circ F^t(x)\,dt \quad \text{and} \quad \mathrm{avg}_-(x) = \int_{-T}^0 f \circ F^t(x)\,dt$$

exist and are equal.

If Lebesgue measure λ on S is non-ergodic, then there is a continuous and compactly supported f on S (with $B = B(f)$ as above), positive measure subsets $B_-, B_+ \subset B$, and real constants $\kappa_- < \kappa_+$ so that

- $\mathrm{avg}(x) < \kappa_-$ for all $x \in B_-$, and
- $\mathrm{avg}(x) > \kappa_+$ for all $x \in B_+$.

Now choose K to be a compact disk in S_∞ with

$$\lambda_\infty(K) > \max\big(1 - \lambda(B_\pm)\big).$$

Since $A^{t_n}(S) \to S_\infty$, there are embeddings $\epsilon_n : K \hookrightarrow A^{t_n}(S)$ for n sufficiently large.

Let $L = \limsup A^{-t_n} \circ \epsilon_n(K) \subset S$.
Then $\lambda(L) \geq \lambda_\infty(K)$, so there is a $b_- \in L \cap B_-$. So, up to passing to a subsequence, we can assume that

$$A^{t_n}(S, b_-) \to (S_\infty, c_-)$$

for some $c_- \in K \subset S_\infty$.

Again, let $L' = \limsup A^{-t_n} \circ \epsilon_n(K) \subset S$.
There is a $b_+ \in L' \cap B_+$, and up to subsequence

$$A^{t_n}(S, b_-, b_+) \to (S_\infty, c_-, c_+)$$

for some $c_-, c_+ \in K \subset S_\infty$.

End of slides, switch to board...