Cutting and resewing pillowcases

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ICERM- Geometric structures in
low-dimensional dynamics.

18 Nov 2013

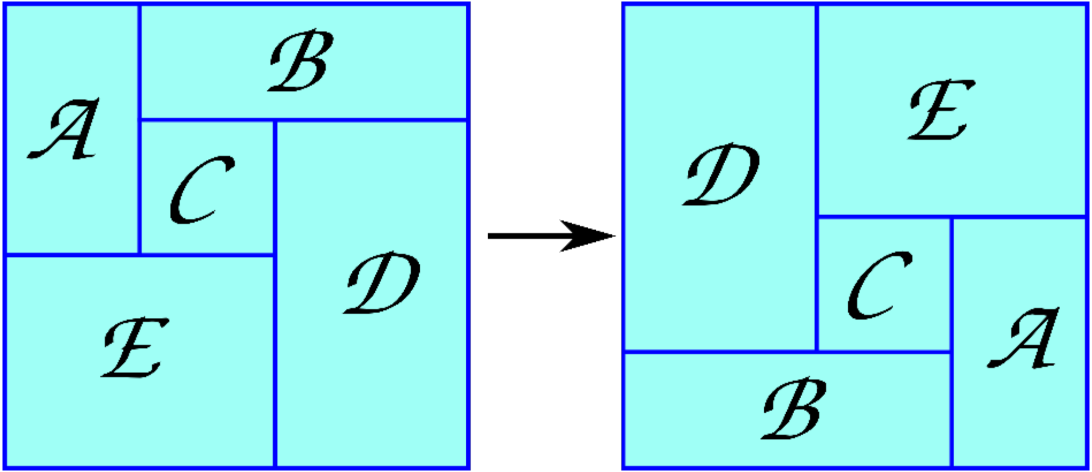
New highbrow title:

Earthquakes on the Riemann sphere paired with a quadratic differential

Outline of talk

- Motivation: The "arithmetic graph"
  - Renormalization
- Piecewise isometries of the square pillowcase
- Invariant fractal curves from substitutions.

The arithmetic graph of a piecewise translation



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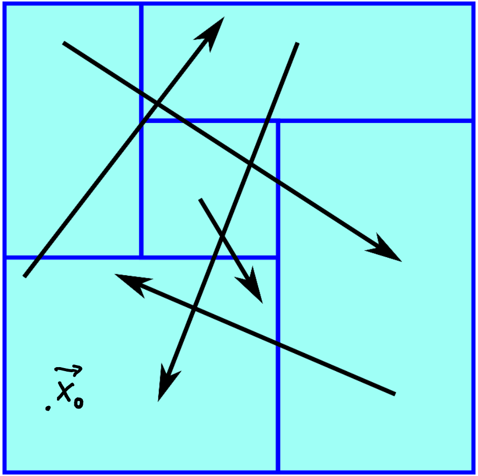
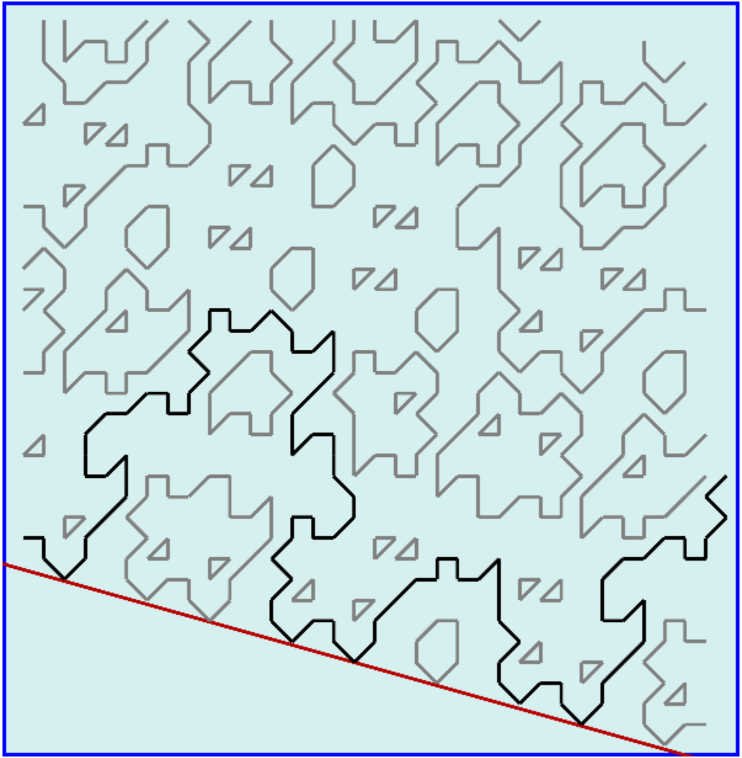


- Let $\vec{v}_1, \dots, \vec{v}_5$ be the translation vectors.
- Additive group: $G = \langle \vec{v}_1, \dots, \vec{v}_5 \rangle$
- Vertices: $\mathcal{V} = (\vec{x}_0 + G) \cap \text{Domain}$
- Edges: $\forall \vec{x} \in \mathcal{V}$ join $\vec{x}$ to $T(\vec{x})$.

Example:

An arithmetic graph arising from outer billiards on a kite.

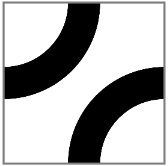
From: Rich Schwartz, "Outer billiards on Kites."



Reversing the construction:
Arithmetic graph → Dynamical System

- The Truchet tiles are:

T_{-1}

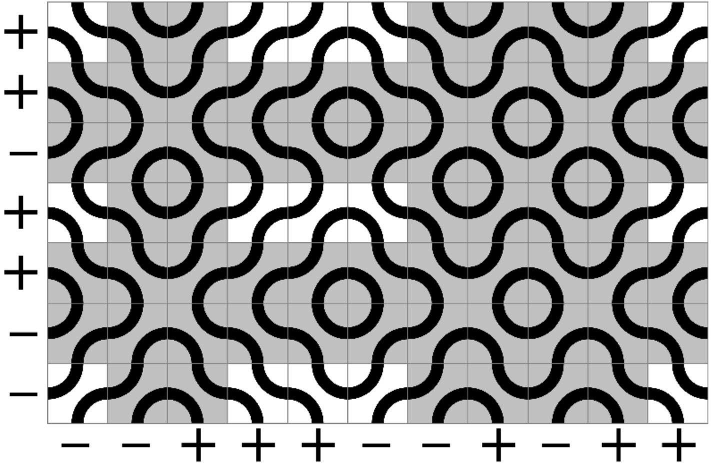


T_1

- A function $f : \mathbb{Z}^2 \to \{\pm 1\}$ gives rise to a tiling.
- A translation invariant collection of tilings gives rise to a dynamical system via "curve following."
- If the tilings are quasi-periodic, this construction can give rise to PETs.

Renormalizable tiling space:

$$T = \left\{ f : \mathbb{Z}^2 \to \{\pm 1\} : \begin{array}{l} \exists\, g, h : \mathbb{Z} \to \{\pm 1\} \\ \text{s.t. } f(m,n) = g(m)h(n) \end{array} \right\}$$



Tilings giving rise to PETs:

Define $\chi : \mathbb{R}/\mathbb{Z} \to \{\pm 1\}$ by $\chi(t) = \begin{cases} 1 & \text{if } t < \frac{1}{2} \\ -1 & \text{if } t \geq \frac{1}{2}. \end{cases}$

Define $g_{\alpha, s_0}(m) = \chi(s_0 + m\alpha)$ and $h_{\beta, t_0}(n) = \chi(t_0 + n\beta)$

Define $\mathcal{T}_{\alpha,\beta} = \{(m,n) \mapsto g_{\alpha,s_0}(m)\, h_{\beta,t_0}(n) : s_0, t_0 \in \mathbb{R}/\mathbb{Z}\}.$

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D = set of four elements

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Dynamical System: $\tau_{\alpha,\beta} : \mathcal{T}_{\alpha,\beta} \times D \circlearrowleft$

$D =$ set of four elements

Renormalizable:

There is a return map of $\tau_{\alpha,\beta}$ which is conjugate to $\tau_{\gamma(\alpha),\gamma(\beta)}$ where γ is the even Gauss map.

Dynamics on the pillowcase:

$\mathcal{T}_{\alpha,\beta} \circlearrowright \mathcal{T}_{\alpha,\beta} \times D$ = Four copies of a tiling space

Dynamics on the pillowcase:

$$
\begin{tikzcd}
\tau_{\alpha,\beta} \circlearrowright & \mathcal{T}_{\alpha,\beta} \times D \arrow[d, "\text{coding rotations}"] & = \text{Four copies of a tiling space} \\
\text{Piecewise translation} \circlearrowright & \mathbb{T}^2 \times D & = \text{Four copies of the torus}
\end{tikzcd}
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Dynamics on the pillowcase:

$\mathcal{T}_{\alpha,\beta} \circlearrowright \mathcal{T}_{\alpha,\beta} \times D$ = Four copies of a tiling space

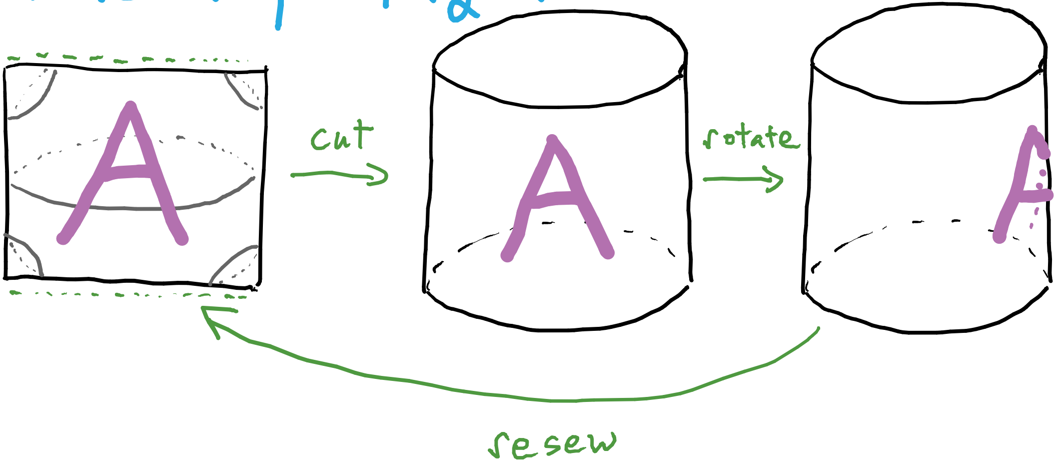
$\big\downarrow$ coding rotations

Piecewise translation $\circlearrowright \mathbb{T}^2 \times D$ = Four copies of the torus

$\big\downarrow$ mod out by symmetries

$T_{\alpha,\beta} \circlearrowright P \times \{h, v\}$ = two copies of the square pillowcase

The map $H_\alpha : P \to P$:



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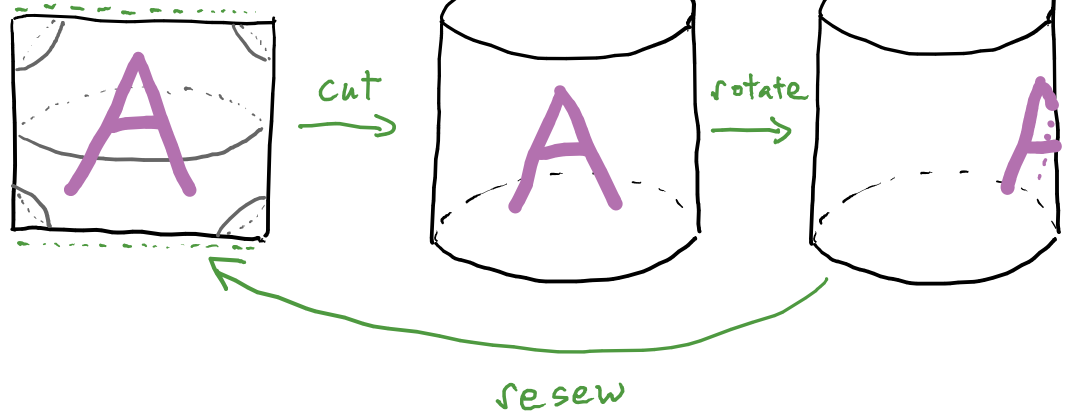

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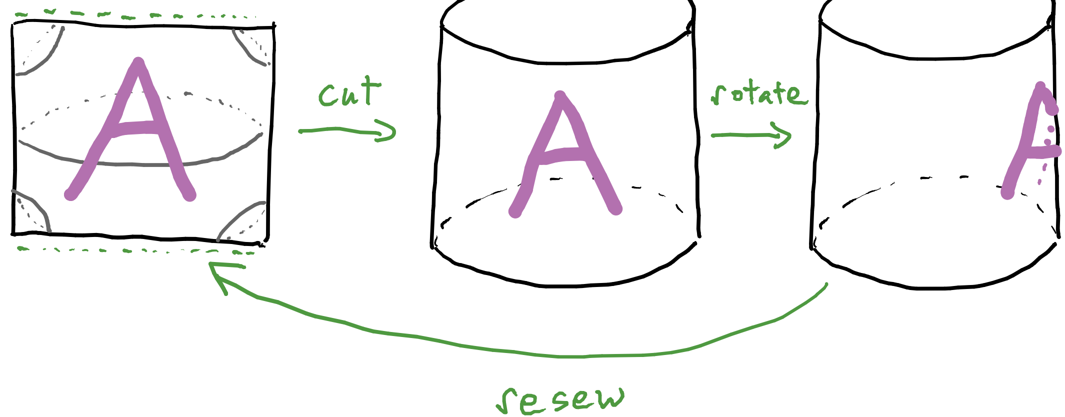
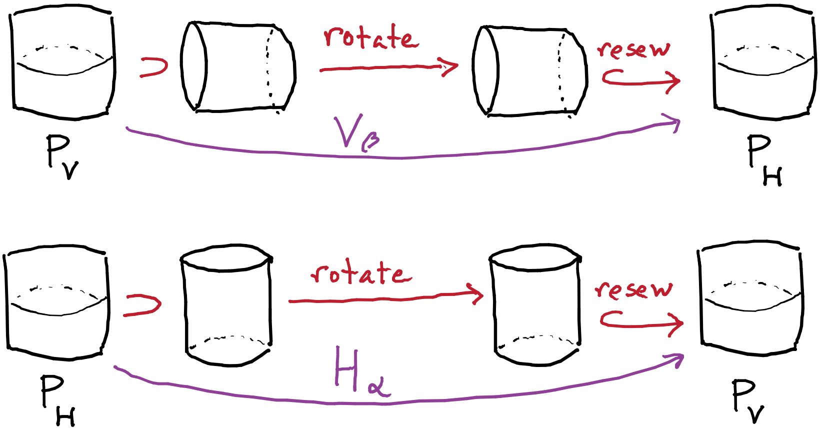


There is also a vertical rotation $V_\beta : P \circlearrowleft$.

We define $S_{\alpha,\beta} = H_\alpha \circ V_\beta : P \circlearrowleft$.

The map we renormalize:

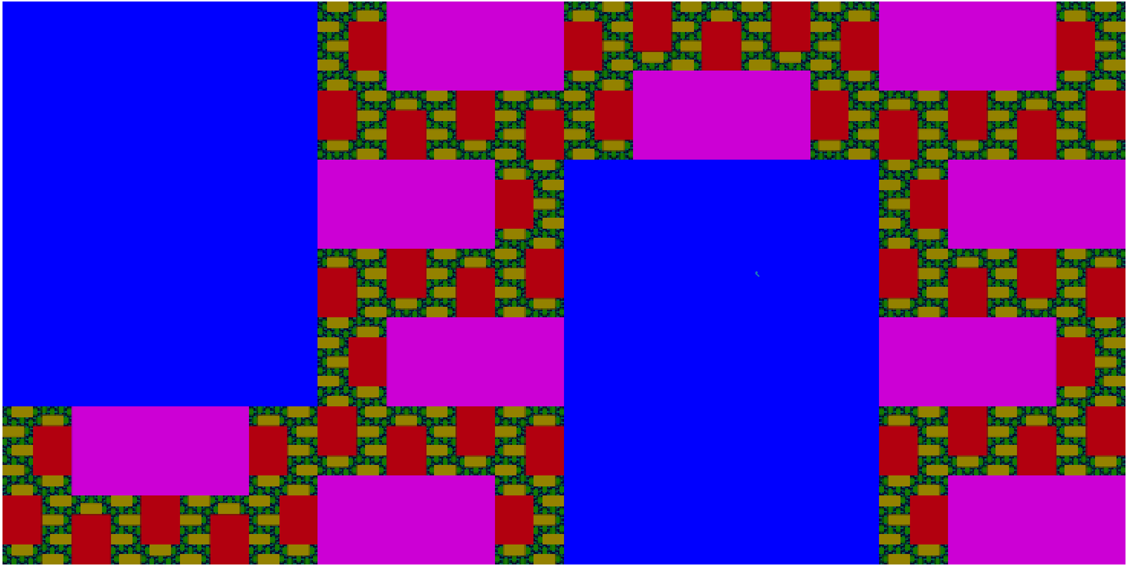
Let P_V and P_H be two copies of the square pillowcase. We define $T_{\alpha,\beta} : P_V \cup P_H \circlearrowleft$ as below:



Example of the map $S_{\alpha,\beta} : P \circlearrowleft$

$$\alpha = \frac{\sqrt{17}-3}{4} \approx 0.28$$

$$\beta = \frac{7-\sqrt{17}}{8} \approx 0.36$$



The aperiodic set:

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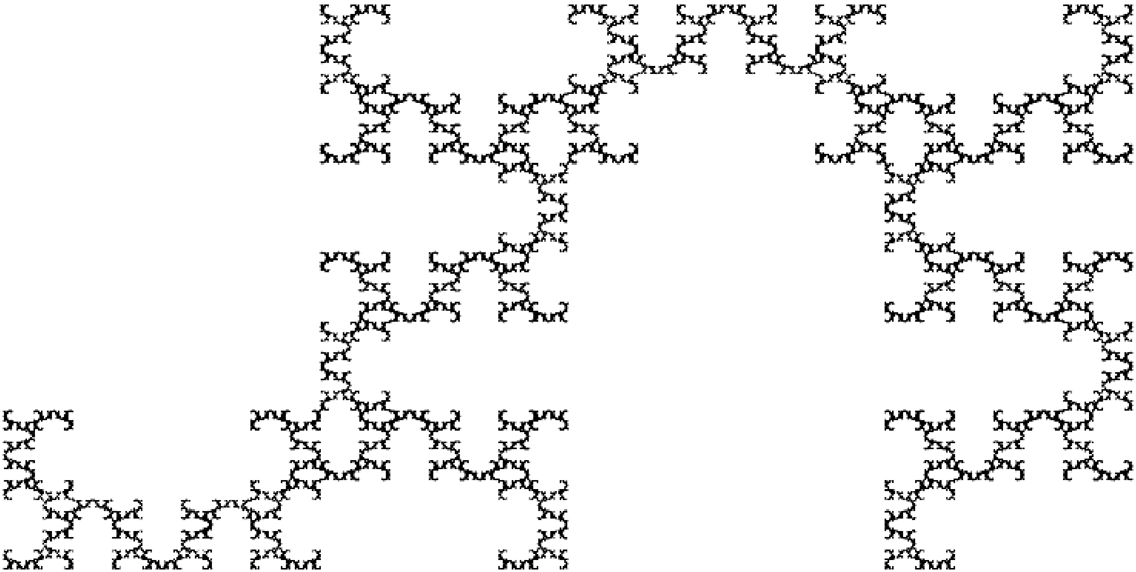
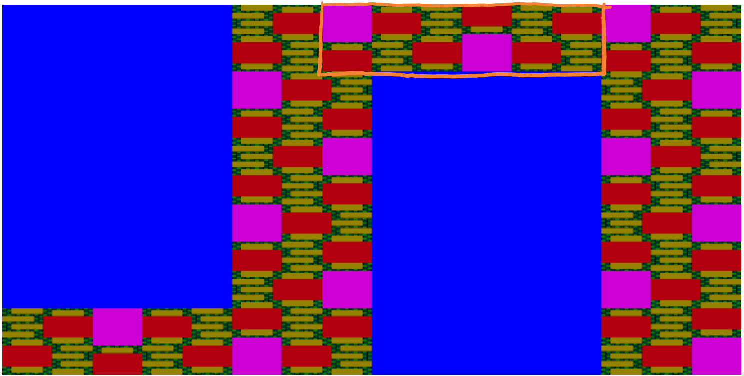
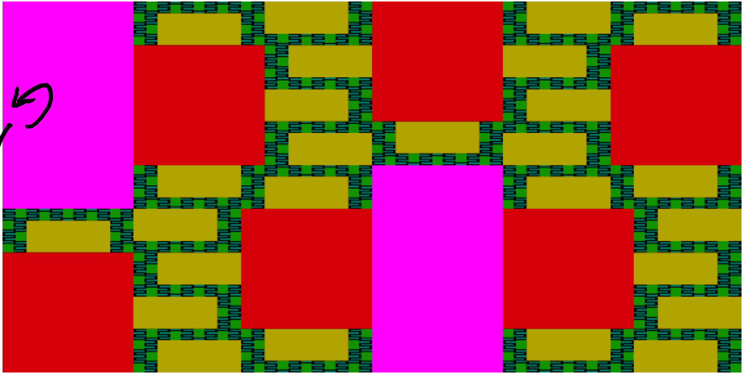


Renormalization:

- $T_{\alpha,\beta} : P_v \cup P_h \circlearrowleft$
- Pick rectangles $R_h \subset P_h$, $R_v \subset P_v$.



- The first return $\hat{T} : R_h \cup R_v \circlearrowleft$ is affinely conjugate to $T_{\gamma(\alpha), \gamma(\beta)}$.



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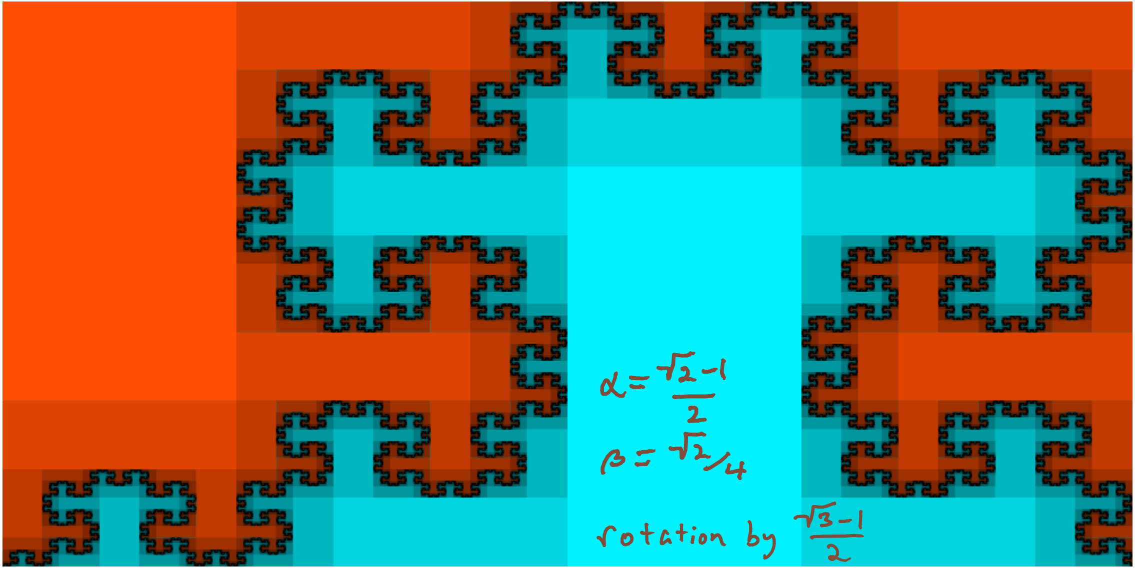
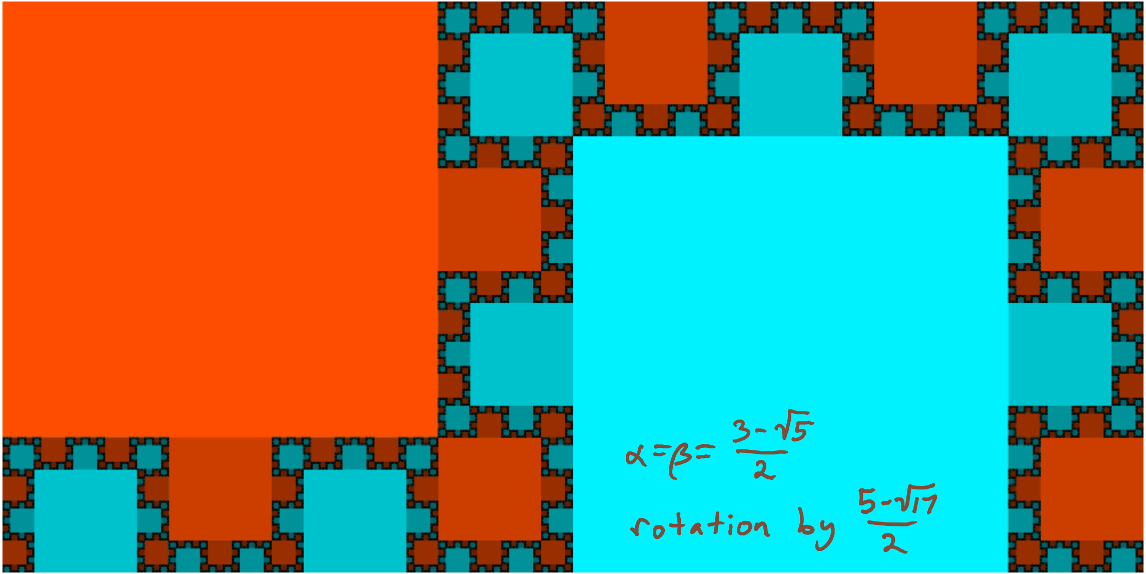
1. $\mu(AP_{\alpha,\beta})=0$ for a.e. (α,β).
2. $\forall \varepsilon>0\ \exists (\alpha,\beta)$ so that $\mu(AP_{\alpha,\beta})>1-\varepsilon$.

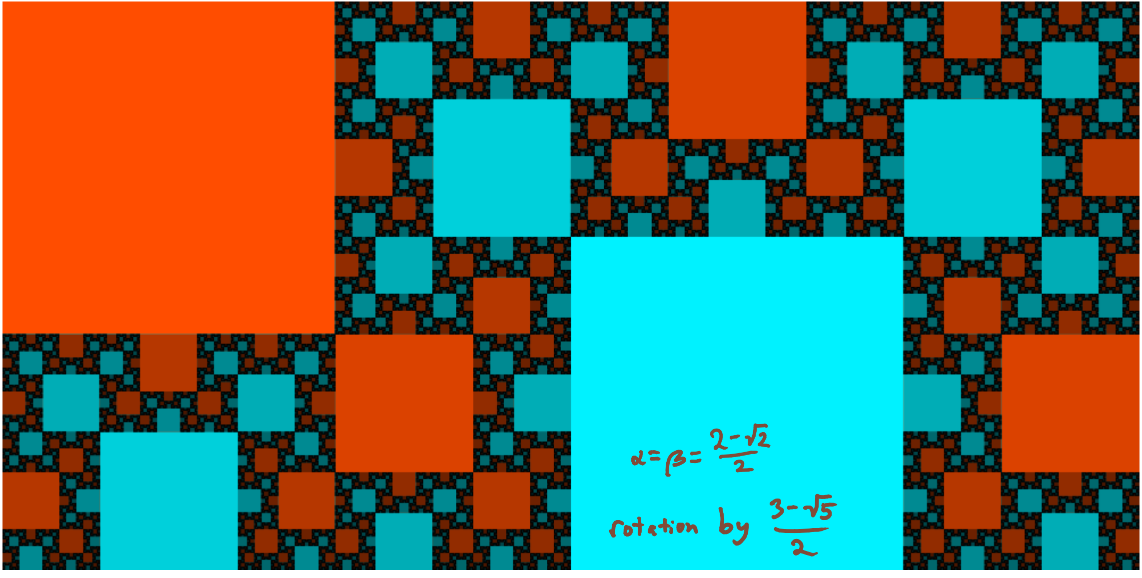
A curve: Sometimes the aperiodic set forms a curve and the restricted dynamics is "conjugate" to a rotation.



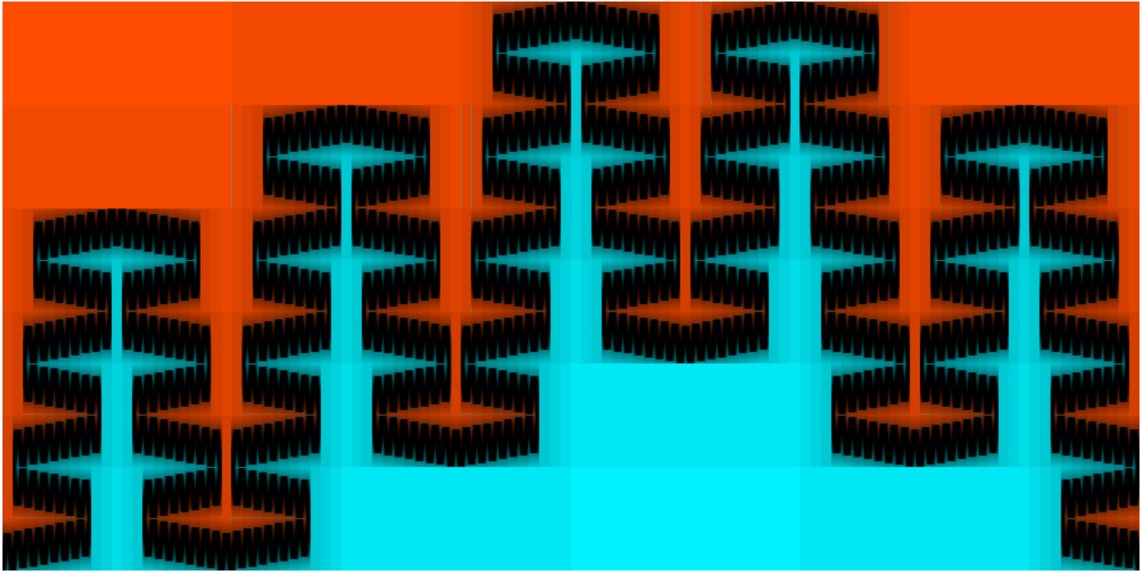
Sometimes the restriction is "semi-conjugate" to a rotation.



Another immersed curve:



Sometimes an embedded curve has positive area.



When do you get a curve?

Pick $\alpha, \beta \in (0, \frac{1}{2}) \setminus \mathbb{Q}$. Let $\langle (m_k, r_k) \rangle_{k \geq 0}$ and $\langle (n_k, s_k) \rangle$ be their even continued fraction expansions.

$$\boxed{m_k, n_k \in \mathbb{Z}_{\geq 0}} \qquad \boxed{r_k, s_k \in \{\pm 1\}}$$

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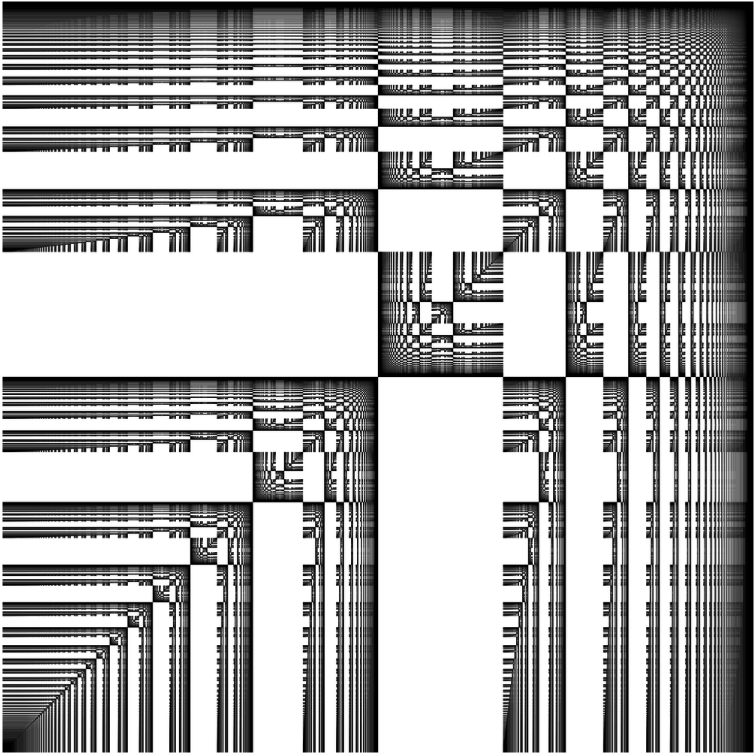
③ The rotation number has even continued fraction sequence $\langle (m_k + n_k, s_k) \rangle$.

Curve parameters

The fractal is the set of pairs (α, β) where

$r_k = s_k \;\; \forall k \geq 0.$



Technicalities:

We have a piecewise isometry $S_{\alpha,\beta} : P \to P$.

We have a rotation $R : \mathbb{R}/\mathbb{Z} \circlearrowleft$.

We have a continuous $\phi : \mathbb{R}/\mathbb{Z} \to P$ s.t. $\phi(\mathbb{R}/\mathbb{Z}) = \overline{AP_{\alpha,\beta}}$.

We want $\phi \circ R(t) = S_{\alpha,\beta} \circ \phi(t)$ $\circledast$.

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1. $(\circledast)$ holds whenever $S_{\alpha,\beta} \circ \phi$ is continuous at t.

2. $S_{\alpha,\beta} \circ \phi(t)$ is continuous on a dense subset of $\mathbb{R}/\mathbb{Z}$.

3. If $S_{\alpha,\beta} \circ \phi$ is not continuous at t, then for any $\langle t_n \rangle \to t$ of points of continuity, $\phi \circ R(t) = \lim\limits_{n \to \infty} S_{\alpha,\beta} \circ \phi(t_n)$.

Outline of proof of semiconjugacy.

I'll sketch the proof of the following:

Suppose the even continued fraction sequences of α and β are $\langle(m_k, r_k)\rangle$ and $\langle(n_k, s_k)\rangle$. If $s_k = r_k = 1$ $\forall k$, then $\exists$ isometry R and continuous ϕ so that:

$$\begin{tikzcd} \Pi_v \cup \Pi_h \arrow[r, "R"] \arrow[d, "\phi"'] & \Pi_v \cup \Pi_h \arrow[d] \\ P_v \cup P_h \supset \overline{AP_{\alpha,\beta}} \arrow[r, "T_{\alpha,\beta}"] & \overline{AP_{\alpha,\beta}} \end{tikzcd}$$

Outline of proof of semiconjugacy.

1. Finite dynamical systems.
2. Substitutions and coding rotations.
3. Coding rectangle exchanges with the same substitutions.

Replacing subwords:

Let $\mathcal{L} = \{A, B, C, \ldots, H\}$.

Form the monoid $\mathcal{L}^*$.

Let $w_1, w_2, w_3, w_4 \in \mathcal{L}^*$.

We write $w_1 \xrightarrow{w_3 \to w_4} w_2$ if

$\exists\, w_-, w_+ \in \mathcal{L}^*$ so that

$$w_1 = w_- w_3 w_+ \quad \text{and} \quad w_2 = w_- w_4 w_+.$$

Begetting Quadruples.

Definition: $Q = (w_1, w_2, w_3, w_4) \in (\mathcal{L}^*)^4$ is *begetting* if

$$w_2 w_1 \xrightarrow{DA \to EF} * \xrightarrow{BC \to GH} w_3 w_4 \quad \text{and}$$

$$w_4 w_3 \xrightarrow{HE \to AB} * \xrightarrow{FG \to CD} w_1 w_2 .$$

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<u>Favorite</u> <u>Example</u>: $Q_0 = (ABC, D, EFG, H)$.

$$DABC \xrightarrow{DA \to EF} EFBC \xrightarrow{BC \to GH} EFGH$$

$$HEFG \xrightarrow{HE \to AB} ABFG \xrightarrow{FG \to CD} ABCD$$

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As a finite dynamical system:

$$Q = (HDABCDH,\ D,\ DHEFGHD,\ H)$$

$$w_1 w_2 \xrightarrow[\text{permutation}]{\text{cyclic}} w_2 w_1 = DH\underline{DA}\underline{BC}DH \xrightarrow[\text{replacement}]{\text{minor}} DH\underline{EF}\underline{GH}DH = w_3 w_4$$

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Compatible substitutions

A $\Phi : \mathcal{L}^* \circlearrowleft$ is compatible if

$\Phi(DA) \xrightarrow{DA \to EF} \Phi(EF)$, $\Phi(BC) \xrightarrow{BC \to GH} \Phi(GH)$

$\Phi(HE) \xrightarrow{HE \to AB} \Phi(AB)$ and $\Phi(FG) \xrightarrow{FG \to CD} \Phi(CD)$.

The compatible substitution $\Phi_{m,n}$

<u>Proposition</u> For all $m \geq 0$ and $n \geq 0$, the following is a compatible substitution:

- $\Phi(A) = H(EFGH)^m DA$
- $\Phi(B) = (BCDA)^n B$
- $\Phi(C) = CDH(EFGH)^m$
- $\Phi(D) = D(ABCD)^n$
- $\Phi(E) = D(ABCD)^n HE$
- $\Phi(F) = (FGHE)^m F$
- $\Phi(G) = GHD(ABCD)^n$
- $\Phi(H) = H(EFGH)^m$

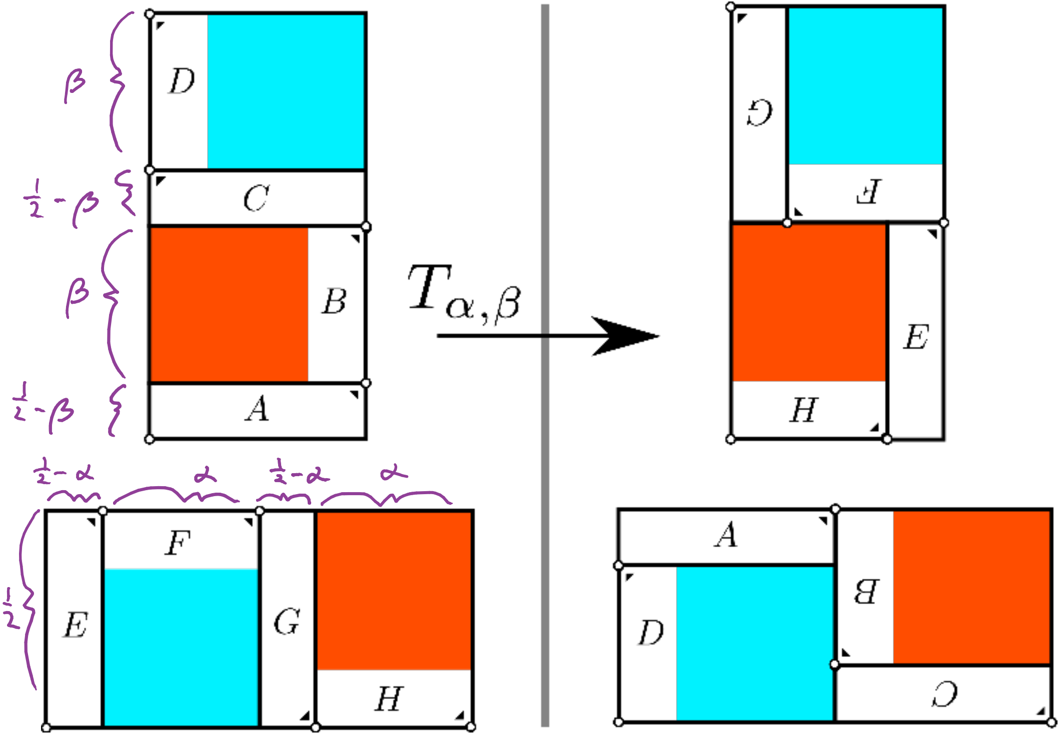
Coding an isometry of two circles.

<u>Thm</u> Let $Q_0 = (ABC, D, EFG, H)$.

Let $\langle m_K \rangle$ and $\langle n_K \rangle$ be sequences of non-negative integers which are not eventually zero. Then the limit of finite dynamical systems associated to

$$\Phi_{m_K, n_K} \circ \cdots \circ \Phi_{m_0, n_0}(Q_0)$$

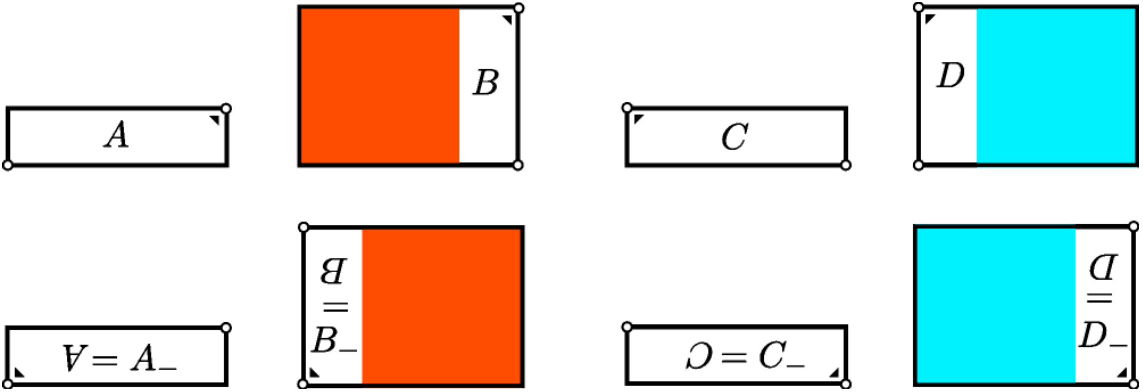
codes a rotation of a pair of circles.



β
D
½ - β
C
β
B
½ - β
A
$T_{\alpha,\beta}$
½ - α
α
½ - α
α
½
E
F
G
H

Flipped rectangles:

For each letter $L \in \{A, \dots, H\}$, we also define a flipped decorated rectangle $R(L_-)$.



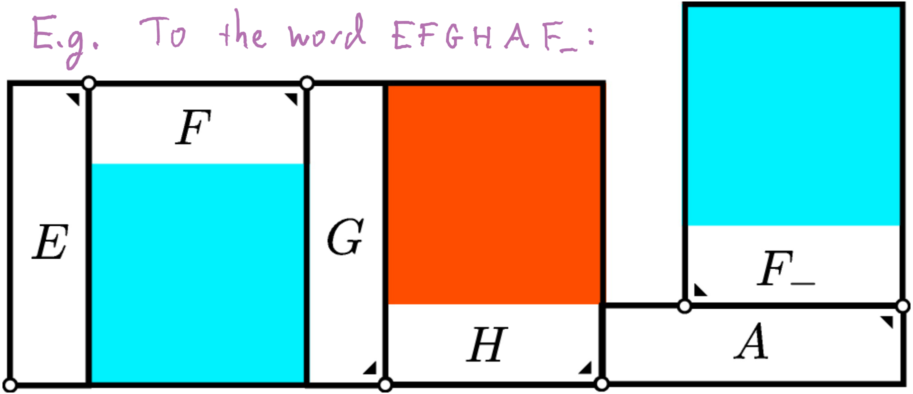
Chains of decorated rectangles:

To each word in the alphabet

$$\mathcal{L}_{\pm} = \{A, \dots, H\} \cup \{A_-, \dots, H_-\}$$

we associate a sequence of decorated rectangles.

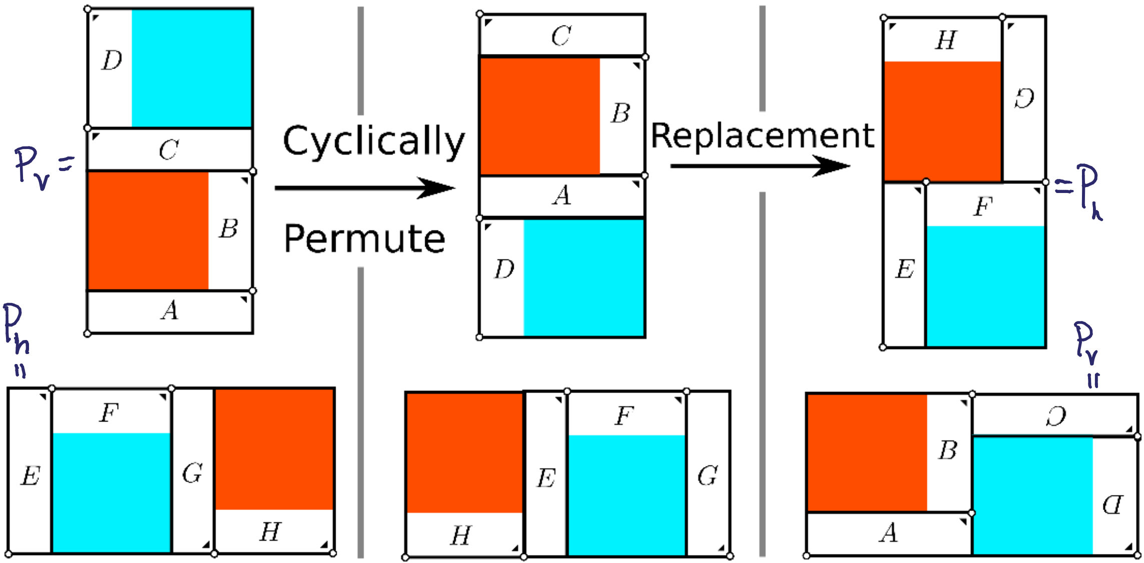
E.g. To the word EFGHAF_:



Alternate view of $T_{\alpha,\beta}$

$DA \to EF$
$BC \to G\_H\_$

$HE \to AB$
$FG \to C\_D\_$



Revisiting the substitutions.

For each $m \geq 0$ and $n \geq 0$, there is a substitution $\Psi_{m,n} : \mathcal{L}^*_\pm \to \mathcal{L}^*_\pm$ so that

1. $\Psi_{m,n}$ commutes with negation, $\neg : \mathcal{L}^*_\pm \to \mathcal{L}^*_\pm$.

2. $\Psi_{m,n}$ extends $\Phi_{m,n}$ to include signs.

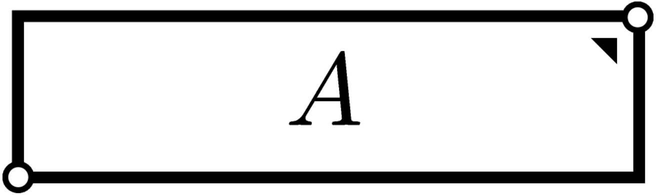
3. $\Psi_{m,n}$ satisfies the signed replacements:

$$\Psi(DA) \xrightarrow{DA \to EF} \Psi(EF) \qquad \Psi(HE) \xrightarrow{HE \to AB} \Psi(AB)$$

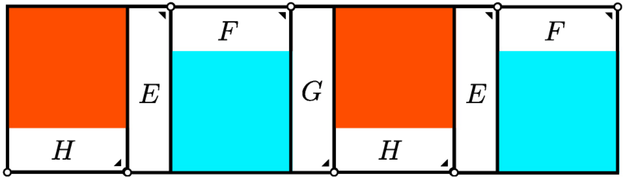
$$\Psi(BC) \xrightarrow{BC \to G_-H_-} \Psi(G_-H_-) \qquad \Psi(FG) \xrightarrow{FG \to C_-D_-} \Psi(C_-D_-).$$

4. Suppose $\gamma(\alpha)=\frac{\alpha}{1-2\alpha}-m$ and $\gamma(\beta)=\frac{\beta}{1-2\beta}-n$.

For each rectangle indexed by $L\in\mathcal{L}$, the chain associated to $\Psi_{m,n}(L)$ with parameters $\alpha'=\gamma(\alpha)$ and $\beta'=\gamma(\beta)$ scaled by $\begin{bmatrix}1-2\alpha & 0\\ 0 & 1-2\beta\end{bmatrix}$

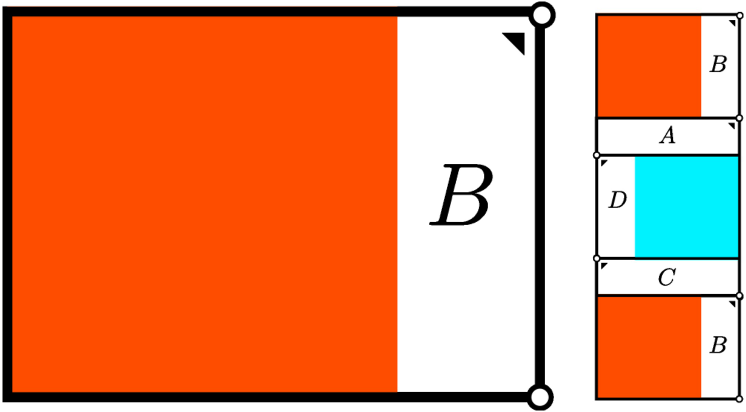


fills the aperiodic subrectangle.

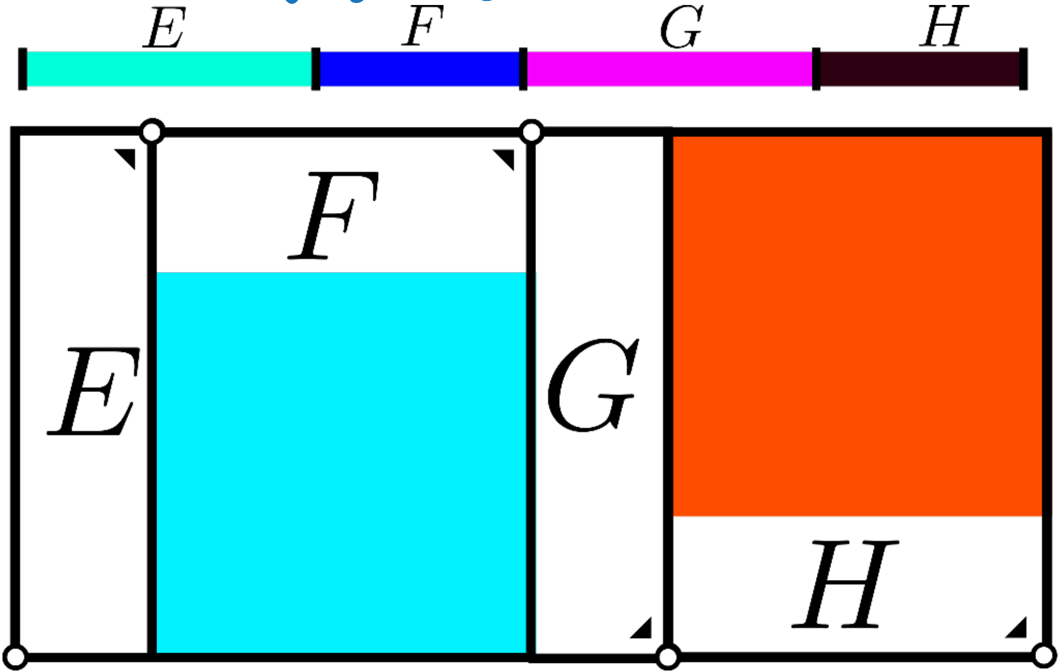


$\Psi(A)=HEFGHEF$

4. Suppose $\gamma(\alpha)=\frac{\alpha}{1-2\alpha}-m$ and $\gamma(\beta)=\frac{\beta}{1-2\beta}-n$. For each rectangle indexed by $L\in\mathcal{L}$, the chain associated to $\Psi_{m,n}(L)$ with parameters $\alpha'=\gamma(\alpha)$ and $\beta'=\gamma(\beta)$ scaled by $\begin{bmatrix}1-2\alpha & 0\\ 0 & 1-2\beta\end{bmatrix}$ fills the aperiodic sub rectangle.



The conjugating map.



The conjugating map.