Cutting and Resewing Pillowcases

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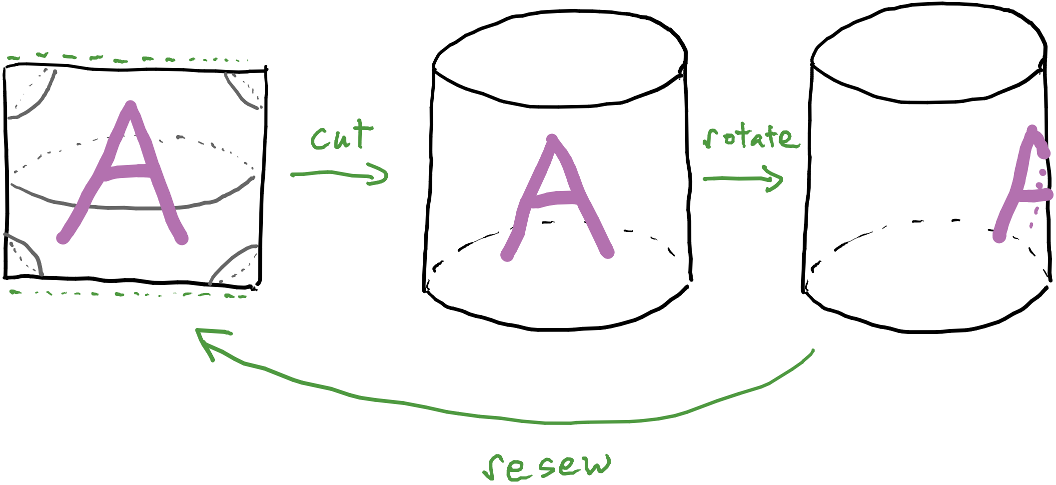
Outline

I. How to cut and resew pillowcases.

II. Theorems: Measure Theoretic + Topological

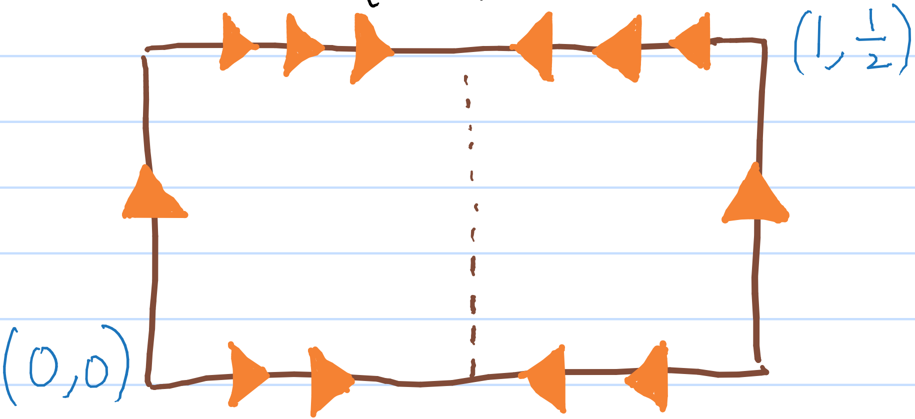
III. Substitutions and Invariant curves.

Piecewise isometries on the square pillowcase



Definition of the pillowcase.

- Let $G < \mathrm{Isom}(\mathbb{R}^2)$ be
  $\langle (x,y) \mapsto (x+1,y),\ (x,y) \mapsto (x,y+1),\ (x,y) \mapsto (1-x,1-y) \rangle$.
- $P = \mathbb{R}^2/G$ is the square pillowcase.



A piecewise isometry from a cylinder in P.

- $P = \mathbb{R}^2/G$ is the square pillowcase.
- $R = [0,1] \times [0, \frac{1}{2}]$ is a fundamental domain.
- $[0,1] \times (0, \frac{1}{2})/G$ is a cylinder.
- For $\alpha \in \mathbb{R}/\mathbb{Z}$ define $H_\alpha : P \to P$ by $H_\alpha = (T_{\alpha,0} \circ \pi_R)/G$ where π_R is projection $P \to R$ and $T_{\alpha,0}(x,y) = (x+\alpha, y)$.

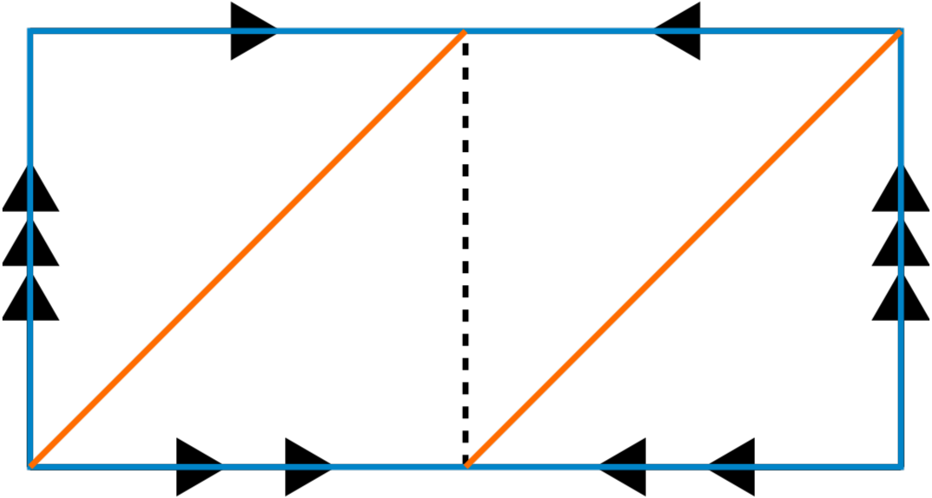
Lots of cylinders

<u>Fact</u> Let θ be a direction of rational slope. Then there are two line segments L_1 and L_2 in the direction θ on the pillowcase P which join pairs of singularities. And

$$P \setminus (L_1 \cup L_2)$$

is a cylinder.

Example: The cylinder in the direction of slope 1:



Piecewise Isometries:

- Take a sequence of rational slopes,
  $\theta_1, \theta_2, \dots, \theta_n$,
  and a sequence of rotation amounts
  $d_1, d_2, \dots, d_n$.
- Let $T_j : P \circlearrowleft$ rotate by d_j in the cylinder associated to θ_j.
- Define $T = T_n \circ T_{n-1} \circ \dots \circ T_1 : P \circlearrowleft$.

Example #1: $S_{\alpha,\beta} = H_\alpha \circ V_\beta$

- V_β rotates in the vertical cylinder by $\beta \pmod 1$.
- H_α rotates in the horizontal cylinder by $\alpha \pmod 1$.

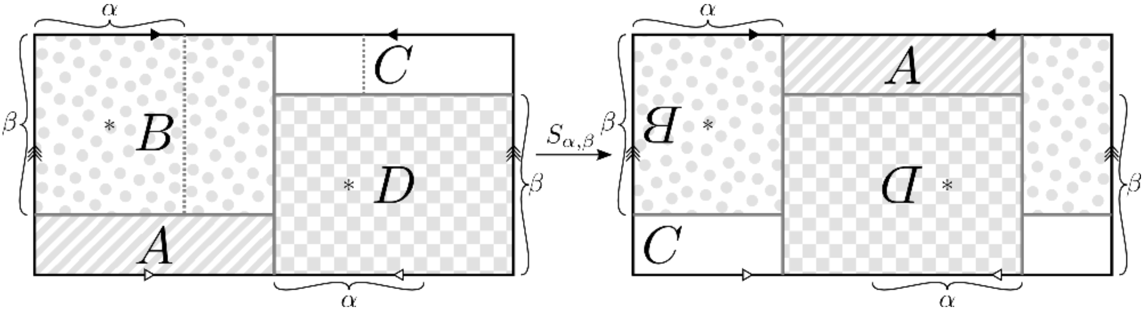


Exhibit A:

$$\alpha = \frac{\sqrt{17}-3}{4} \approx 0.28$$

$$\beta = \frac{7-\sqrt{17}}{8} \approx 0.36$$

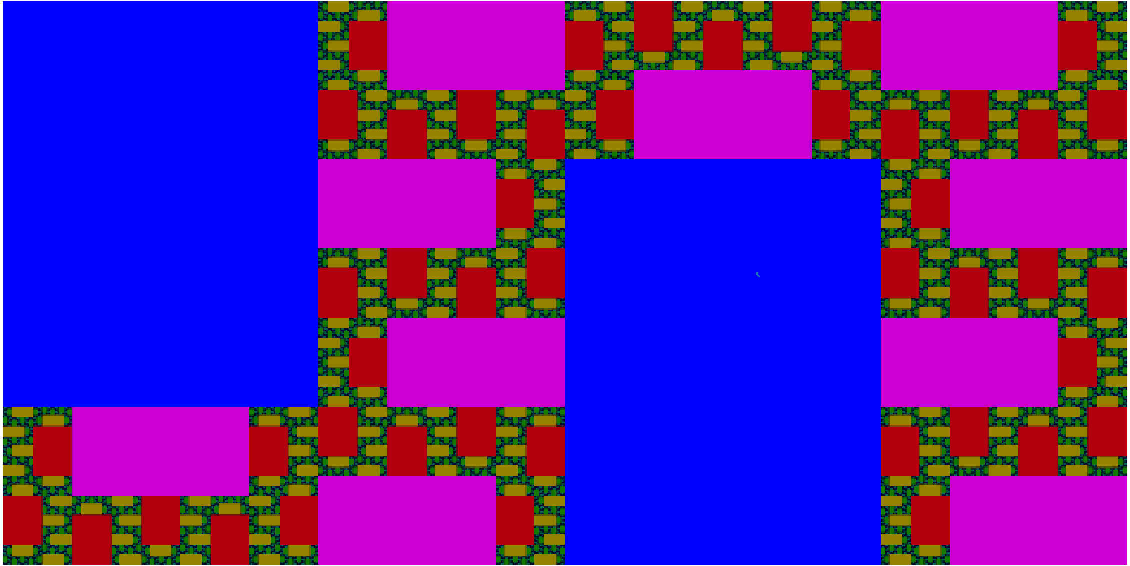


Exhibit A:
The aperiodic set:

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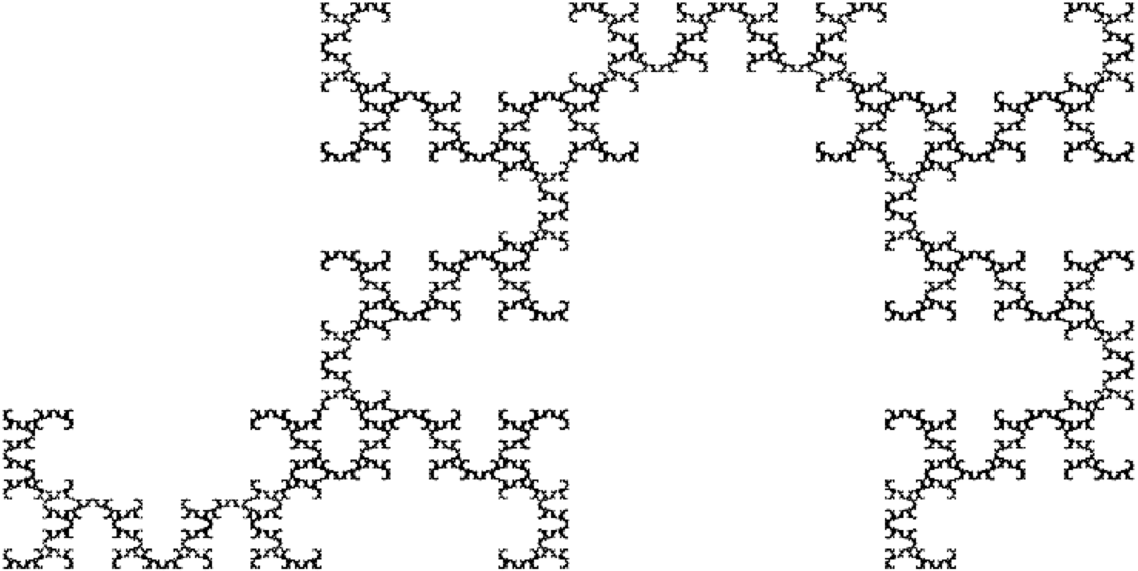

Exhibit B :

$$\alpha = \frac{\sqrt{5}-1}{4} \approx 0.31$$

$$\beta = \frac{3-\sqrt{5}}{4} \approx 0.19$$

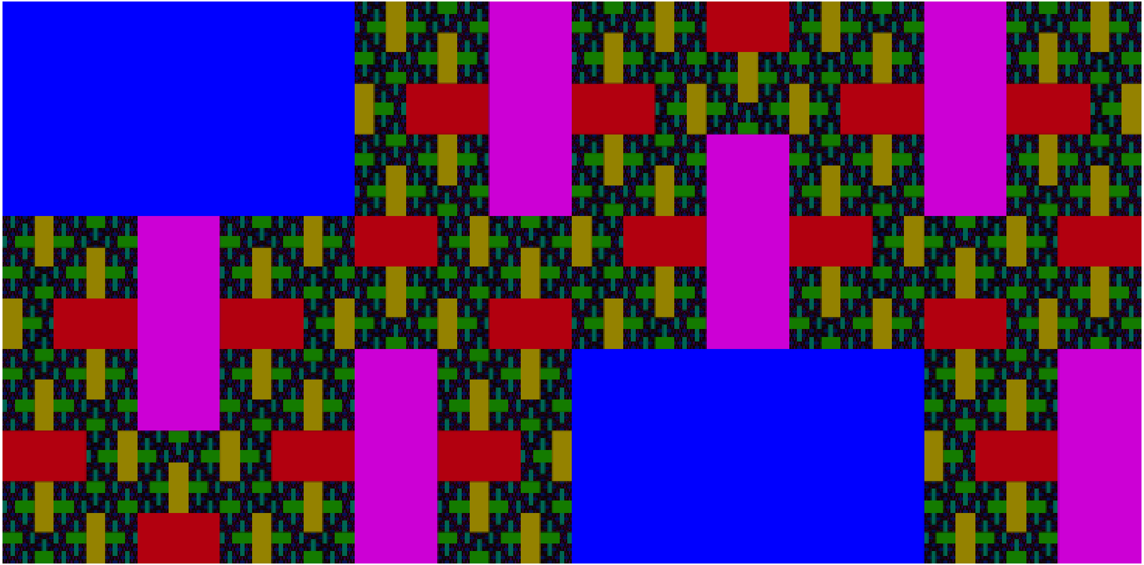

Exhibit B:
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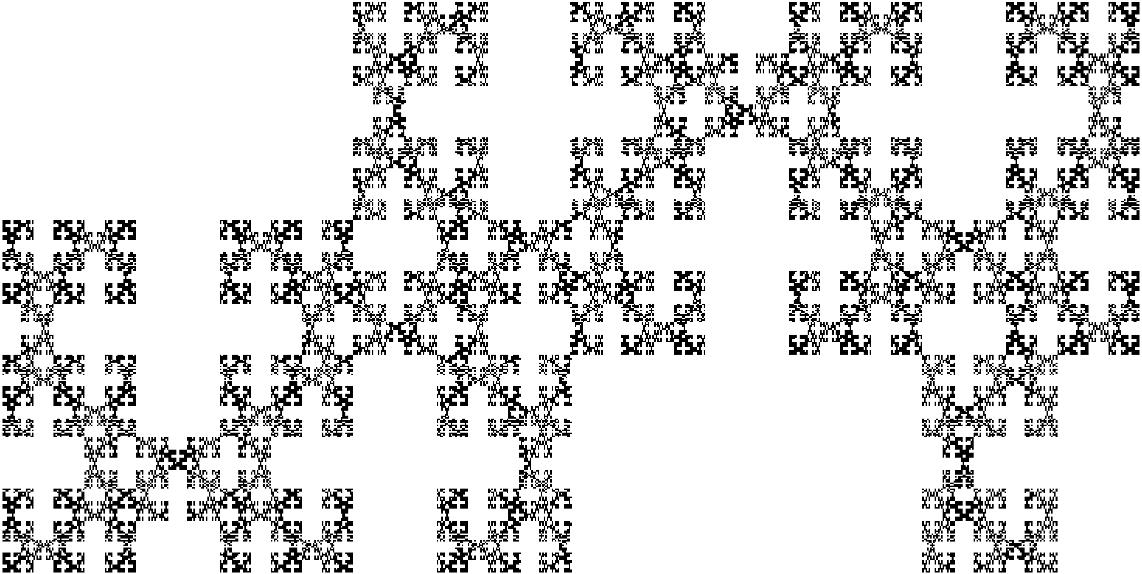


Exhibit C : $\alpha = \beta = \dfrac{\sqrt{17}-3}{4}$

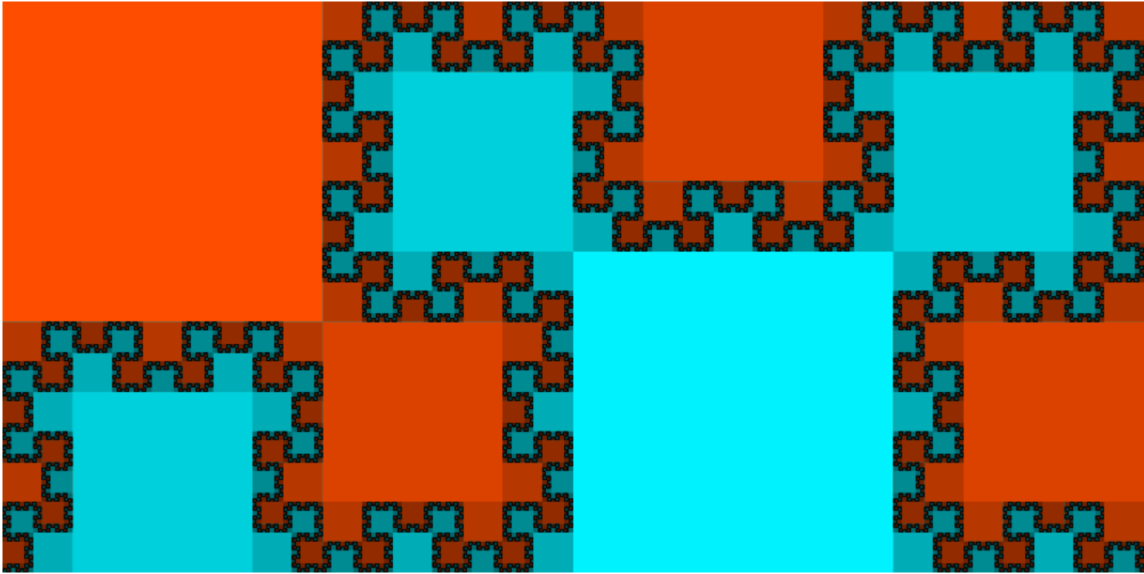
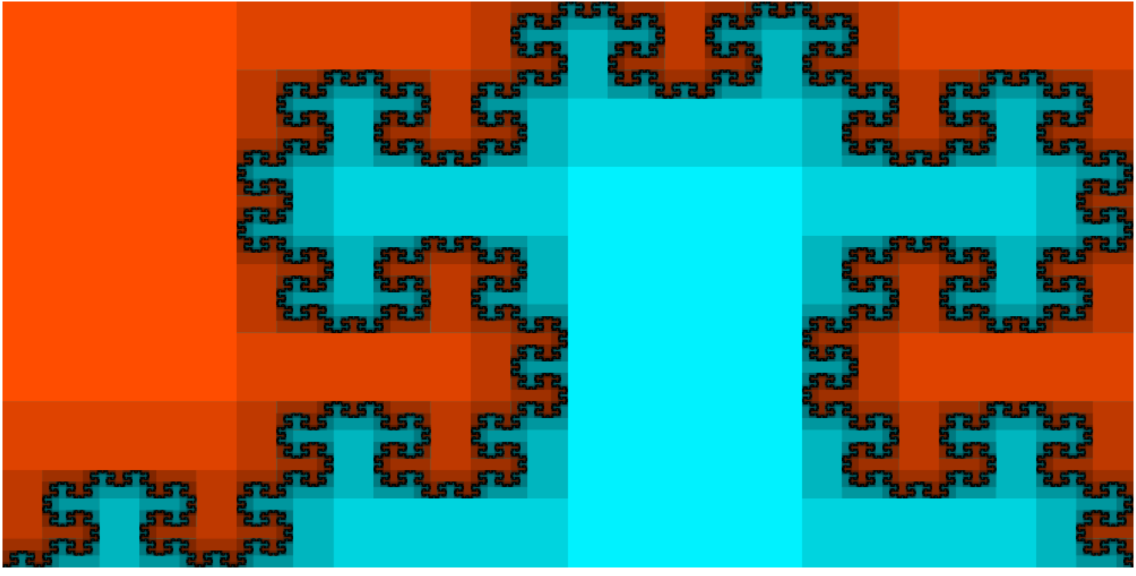


Exhibit D:

$$\alpha = \frac{\sqrt{2}-1}{2} \approx 0.207$$

$$\beta = \frac{\sqrt{2}}{4} \approx 0.354$$



Example #2:

- First rotate in the horizontal cylinder by $\alpha \pmod 1$.
- Then rotate in the vertical cylinder by $\alpha \pmod 1$.
- Then rotate in the slope 1 cylinder by $-\alpha \cdot \sqrt{2} \pmod{\sqrt{2}}$.

Exhibit A:

$$\alpha = \frac{2}{\phi^2}\ ; \quad \phi = \frac{1+\sqrt{5}}{2}.$$

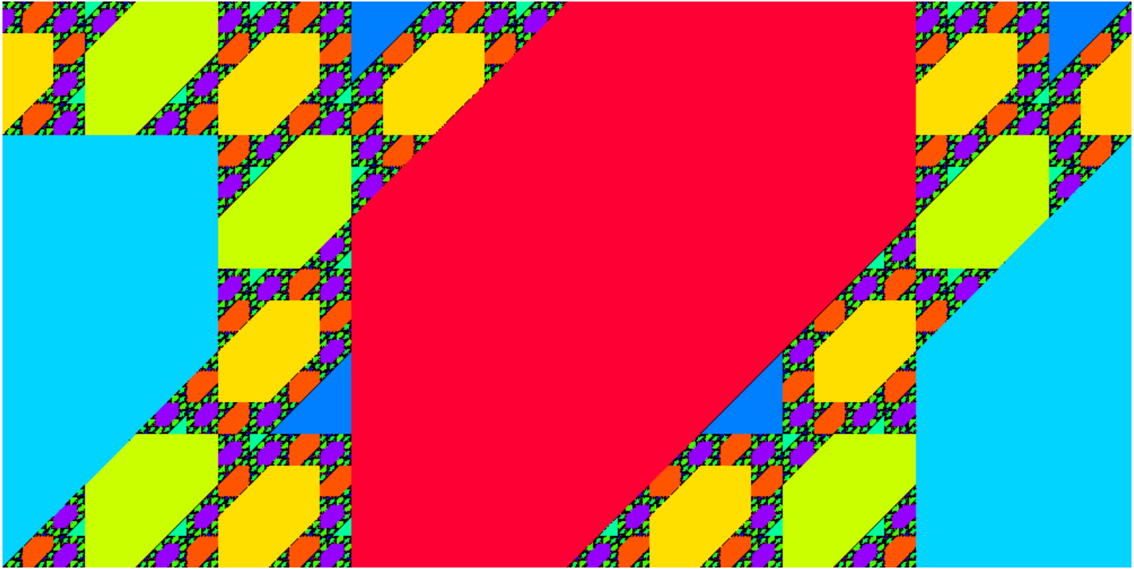
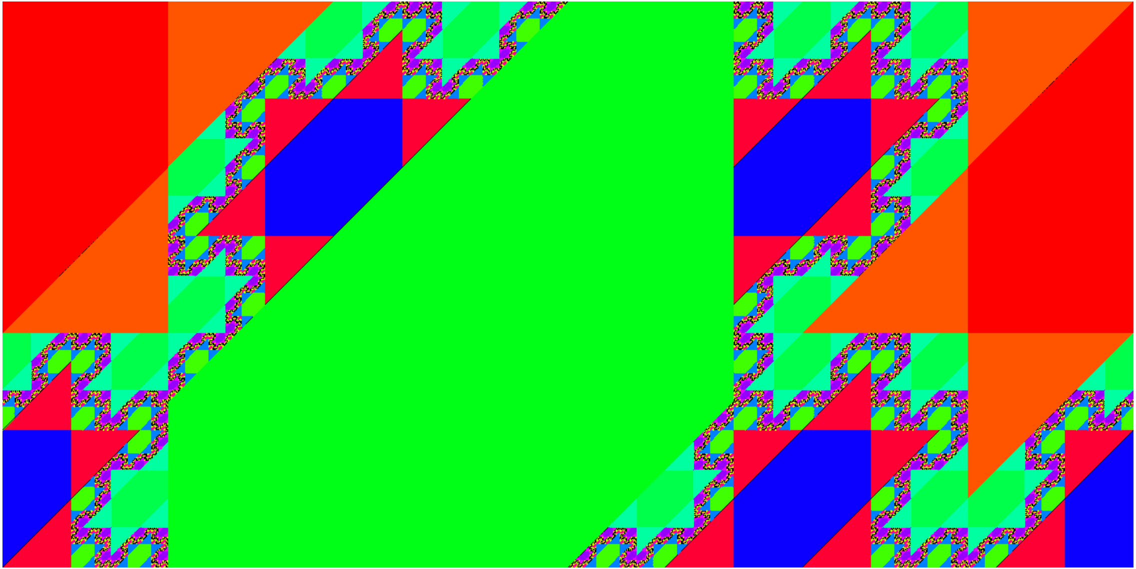


Exhibit B:



Example 3:

Rotate in the horizontal cylinder by $\alpha \pmod{1}$.

Rotate in the slope 1 cylinder by $\frac{1}{4}\sqrt{2} \pmod{\sqrt{2}}$.

Rotate in the vertical cylinder by $\alpha \pmod{1}$.

Rotate in the slope -1 cylinder by $\frac{1}{4}\sqrt{2} \pmod{\sqrt{2}}$.

Exhibit A:

$$\alpha = \frac{\sqrt{2}}{8}$$

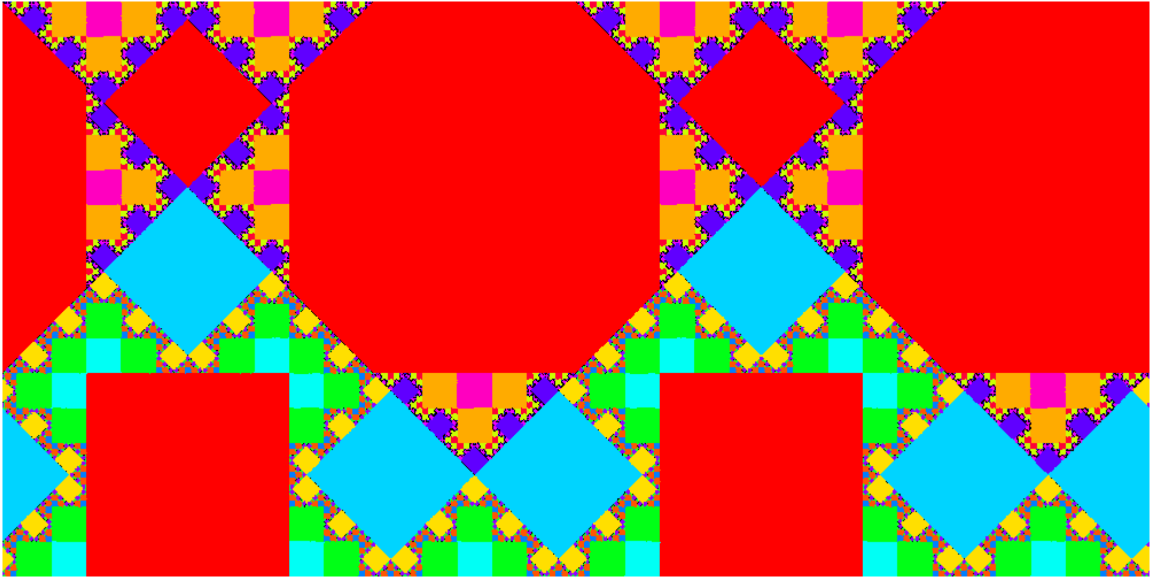
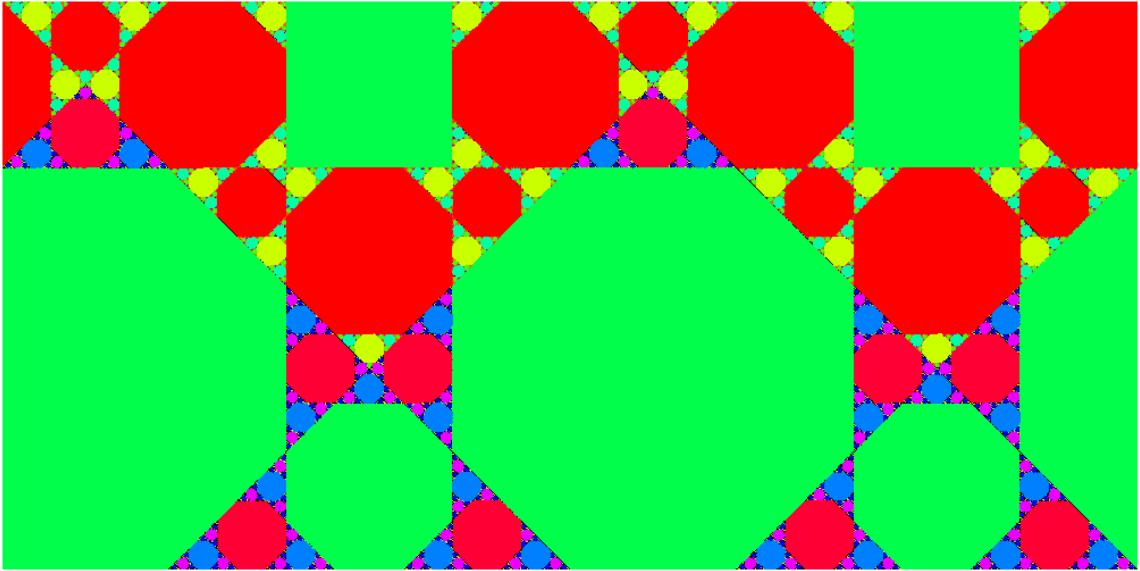


Exhibit B:

$$\alpha = \frac{\sqrt{2}}{4}$$



Connection to outer billiards?

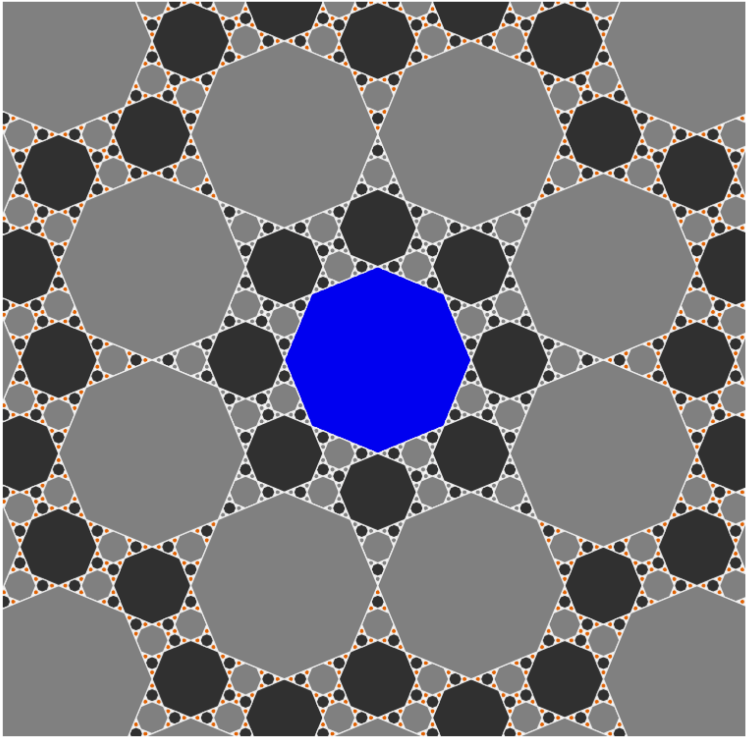


Image from a paper of R. Schwartz.

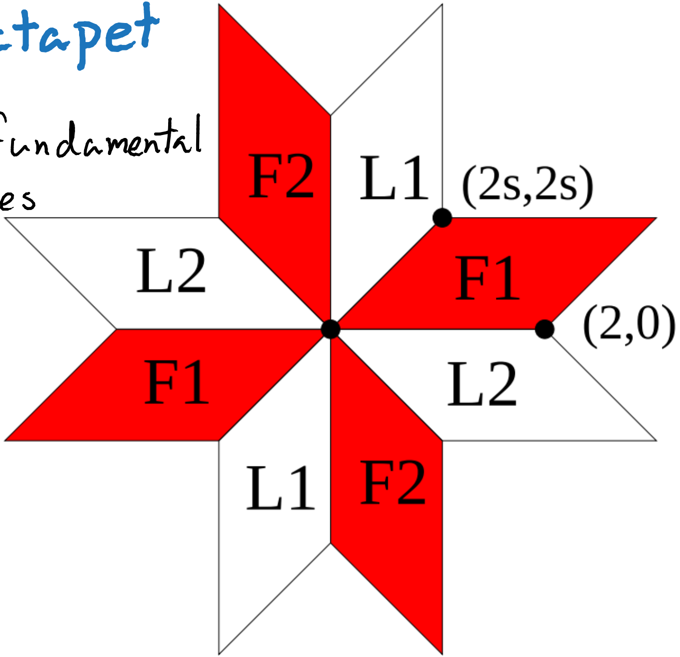
Schwartz's "Octapet"

F_1 and F_2 are fundamental domains for lattices L_1 and L_2.

$x \in F_1 \mapsto x + v \in F_2$ with $v \in L_2$.

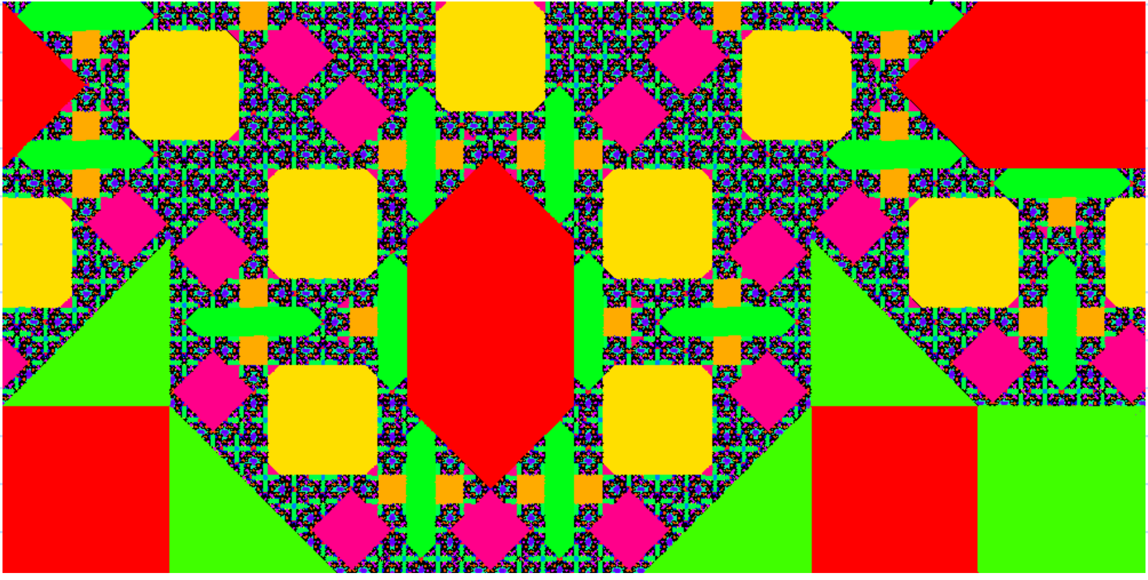
$y \in F_2 \mapsto y + w \in F_1$ with $w \in L_1$.

Image from a paper of R. Schwartz.



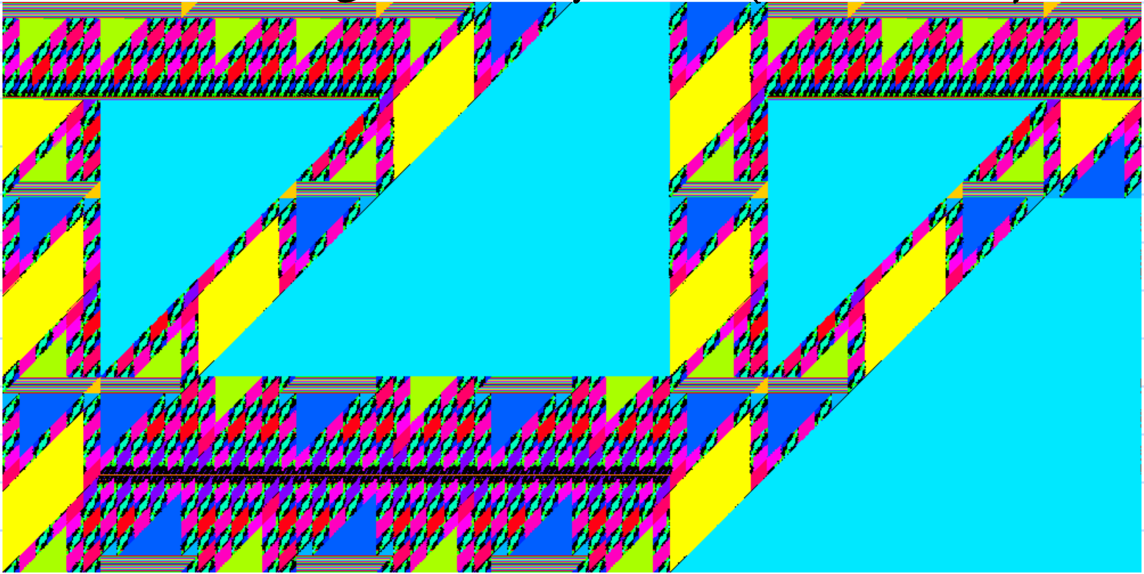
Random Example A:

① Rotate by $\frac{2-\sqrt{2}}{4}$ (mod 1) horizontally.
② Rotate by $\frac{2-\sqrt{2}}{4}$ (mod $\sqrt{2}$) in slope 1 direction.
③ Rotate by $\frac{2-\sqrt{2}}{4}$ (mod 1) vertically.
④ Rotate by $\frac{2-\sqrt{2}}{4}$ (mod $\sqrt{2}$) in slope -1 direction.



Random Example B:

1. Rotate by $\sqrt{2}-1 \pmod 1$ in horizontal.
2. Rotate by $\sqrt{2}-1 \pmod 1$ in vertical.
3. Rotate by $-\sqrt{2}+1 \pmod 1$ in horizontal.
4. Rotate by $2-\sqrt{2} \pmod{\sqrt{2}}$ in slope 1 dir.



Part II: Theorems.

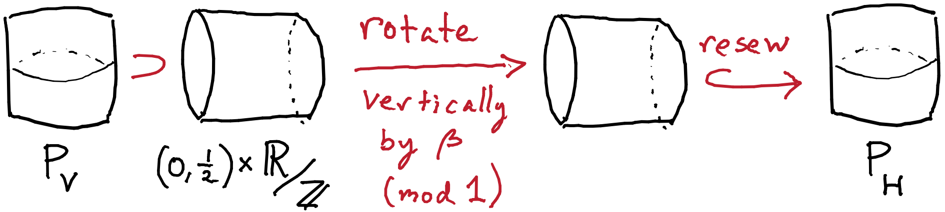
<u>Convention</u>: A piecewise isometry of X is a continuous local isometry from an open dense subset of X to X.

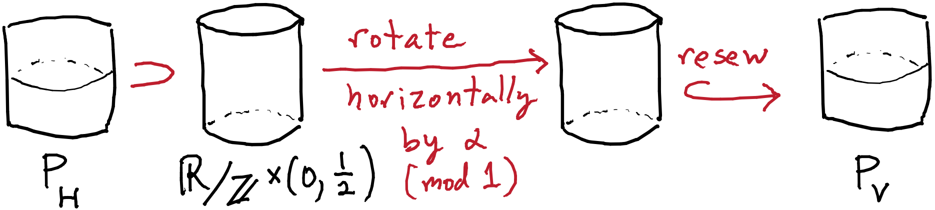
The Maps $S_{\alpha,\beta}$:

- $P = \mathbb{R}^2/G$ is the square pillowcase.
- It contains the cylinders: $C_V = (0,\frac{1}{2}) \times \mathbb{R}/\mathbb{Z}$
  $C_H = \mathbb{R}/\mathbb{Z} \times (0,\frac{1}{2})$.
- Let $\alpha, \beta \in (0,\frac{1}{2})$ be irrational.
- We define isometries $H_\alpha : C_H \circlearrowleft;\ (x,y) \mapsto (x+\alpha, y)$,
  $V_\beta : C_V \circlearrowleft;\ (x,y) \mapsto (x, y+\beta)$.
- We define $S_{\alpha,\beta} = H_\alpha \circ V_\beta : C_V \cap V_\beta^{-1}(C_H) \to P$.

The Map $T_{\alpha,\beta}$:

Let P_V and P_H be two copies of the square pillowcase. We define $T_{\alpha,\beta} : P_V \cup P_H \circlearrowleft$ as below:





Even Continued Fractions*

Let $\mathcal{I}$ denote the group of isometries of $\mathbb{R}$ which fix the set $\mathbb{Z}$.

The even Gauss map $\gamma : (0, \frac{1}{2}) \setminus \mathbb{Q} \circlearrowleft$ is defined by $\gamma(t) = \frac{t}{1-2t} \pmod{\mathcal{I}}$.

Then $\gamma(t) = s\left(\frac{t}{1-2t} - n\right)$ for $n = n(t) \in \mathbb{Z}_{\geq 0}$ and $s = s(t) \in \{\pm 1\}$.

The even continued fraction sequence (ECFS) of t is $\langle (n_k = n \circ \gamma^k(t),\ s_k = s \circ \gamma^k(t)) \rangle$.

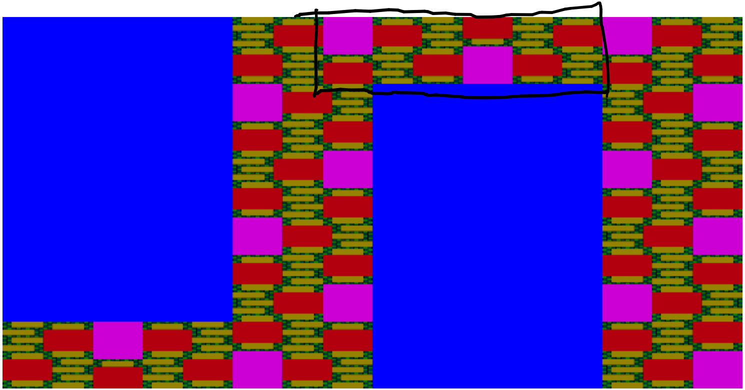
* Our treatment is equivalent but different than the treatment of even continued fractions by Kraaikamp and Lopes.

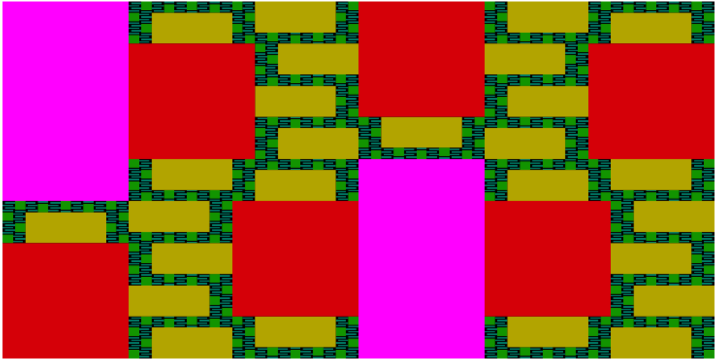
Renormalization.

<u>Thm</u>: Fix irrationals $\alpha, \beta \in (0, \frac{1}{2})$.

There is a pair of rectangular subsets $R_h \subset P_h$ and $R_v \subset P_v$ so that the first return map of $T_{\alpha,\beta} : P_v \cup P_h \circlearrowleft$ to $R_h \cup R_v$ is affinely conjugate to $T_{\gamma(\alpha), \gamma(\beta)}$.

$S_{\alpha,\beta}$





$S_{\gamma(\alpha),\gamma(\beta)}$

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3. If the sequence of pairs $\langle(\gamma^k(\alpha), \gamma^k(\beta))\rangle$ has an accumulation point (x,y) with $x \neq 0$ and $y \neq 0$, then almost every point of P is periodic under $S_{\alpha,\beta}$.

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6. There is a dense set of pairs (α,β) so that the Lebesgue measure of $S_{\alpha,\beta}$-aperiodic points is positive.

A new topological result.

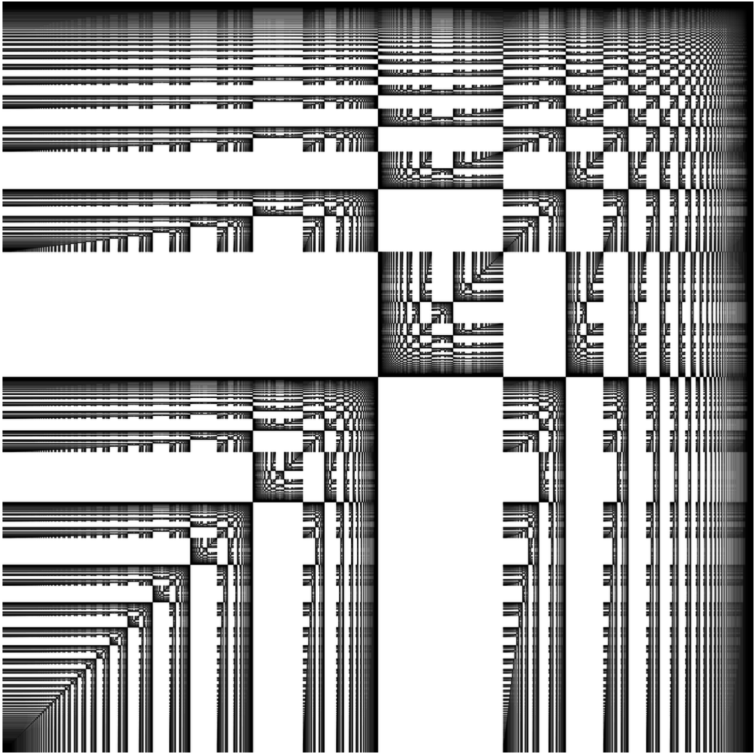
Thm Let $\alpha, \beta \in (0, \frac{1}{2})$ be irrational and let $\langle (m_k, r_k) \rangle_{k \geq 0}$ and $\langle (n_k, s_k) \rangle_{k \geq 0}$ be their even continued fraction expansions.
If $r_k = s_k \in \{\pm 1\}$ for each $k \geq 0$, then there is a continuous surjective map from $\mathbb{R}/\mathbb{Z}$ onto the closure of the set of points in P with defined and aperiodic orbits, and there is a rotation $R : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ so that

$$\phi \circ R(t) = S_{\alpha,\beta} \circ \phi(t) \quad \circledast$$

for all t so that $S_{\alpha,\beta} \circ \phi(t)$ is defined.
Also, $\circledast$ holds for a dense open set of t.

Curve parameters

The set of pairs (α, β) in $(0, \frac{1}{2}) \times (0, \frac{1}{2})$ for which the theorem applies is shown in black at right.



More details:

(1) If the ECFE for α is $\langle (m_k, s_k) \rangle$ and the ECFE for β is $\langle (n_k, s_k) \rangle$, then the rotation R is by the irrational whose ECFE is $\langle (m_k + n_k, s_k) \rangle$.

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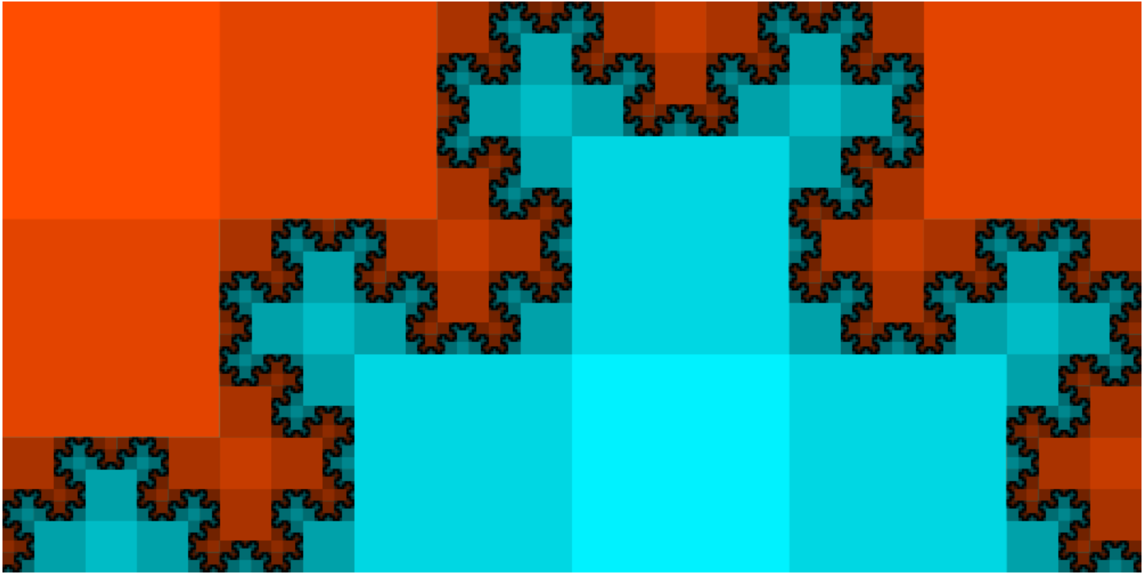
2. If $s_k = 1$ for infinitely many k, then $\phi : \mathbb{R}/\mathbb{Z} \to P$ is an embedding.

3. There are "curve cases" with the measure of a periodic points arbitrarily close to full measure.

An embedded curve example:

$\alpha = \beta = \frac{3-\sqrt{5}}{4} \approx 0.191$

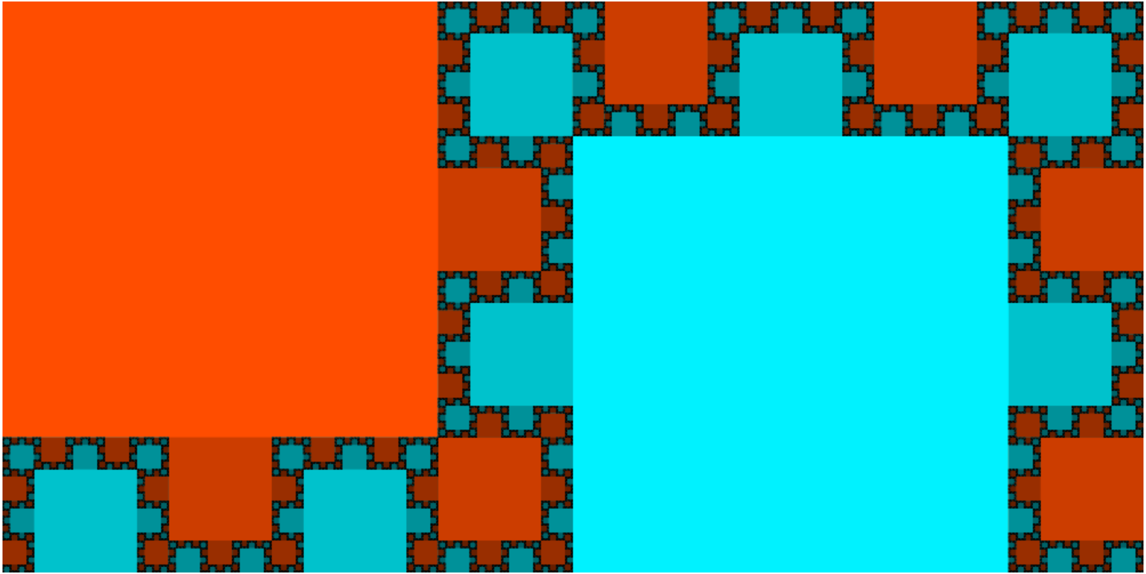
rotation by $\frac{5-\sqrt{17}}{4} \approx 0.219$



An immersed curve:

$\alpha = \beta = \frac{3-\sqrt{5}}{2} \approx 0.382$

rotation by $\frac{5-\sqrt{17}}{2}$



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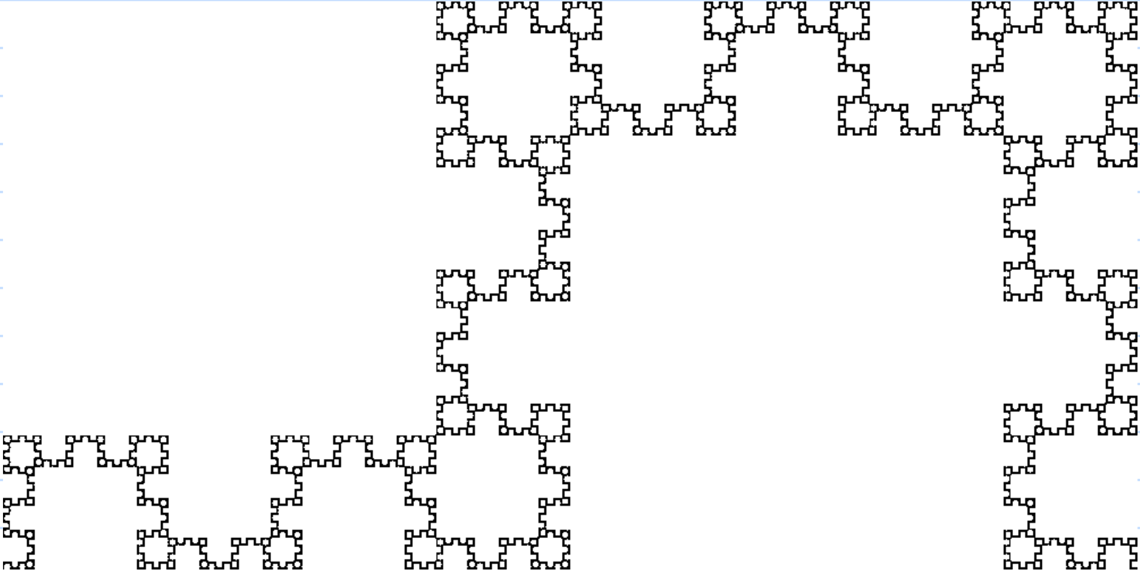
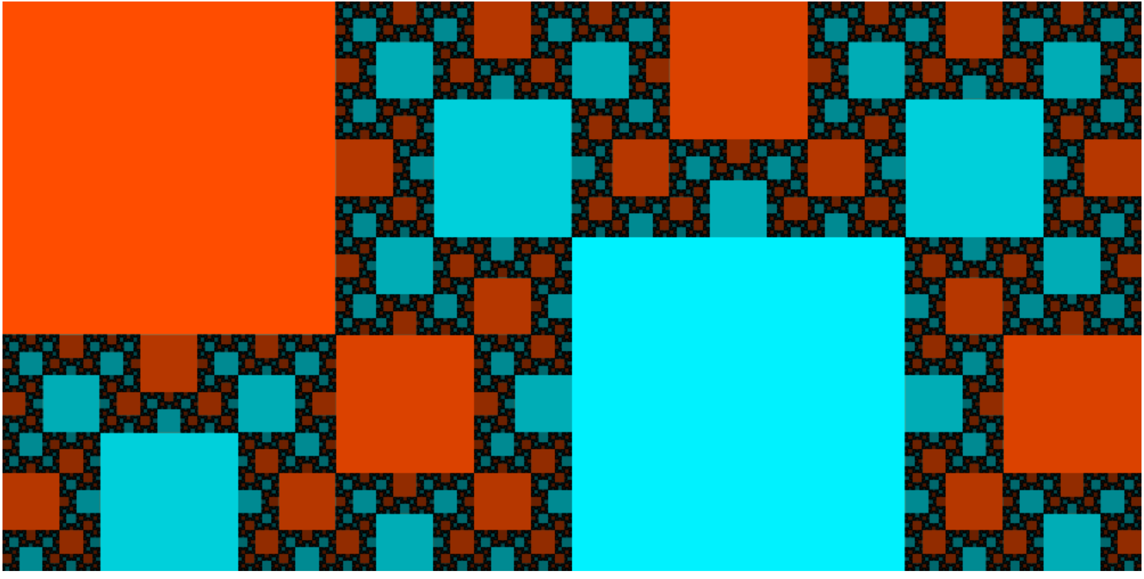


A second immersed curve:

$\alpha = \beta = \frac{2-\sqrt{2}}{2} \approx 0.293$

rotation by $\frac{3-\sqrt{5}}{2} \approx 0.382$



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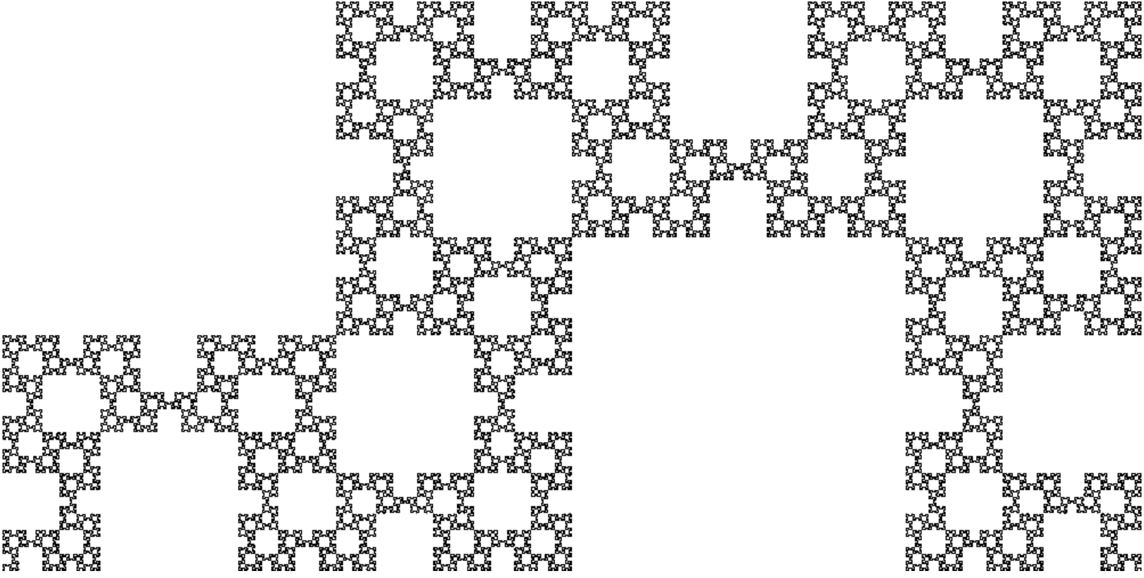
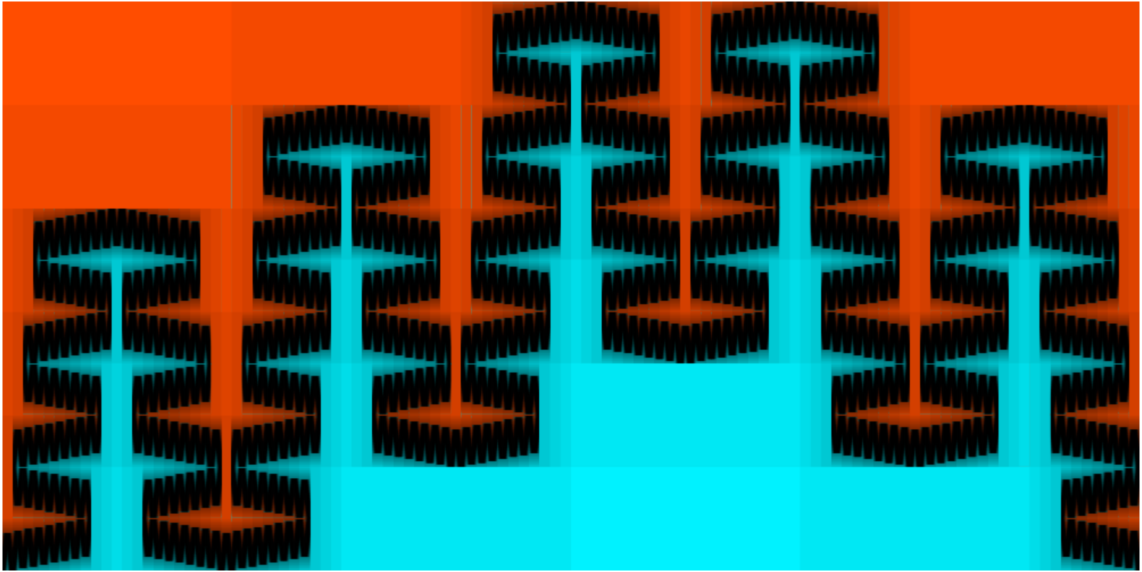


A positive area curve:



Questions for the non-curve case:

<u>Observation</u> If (α, β) is a pair of irrationals not covered by the theorem, there is no "invariant curve."

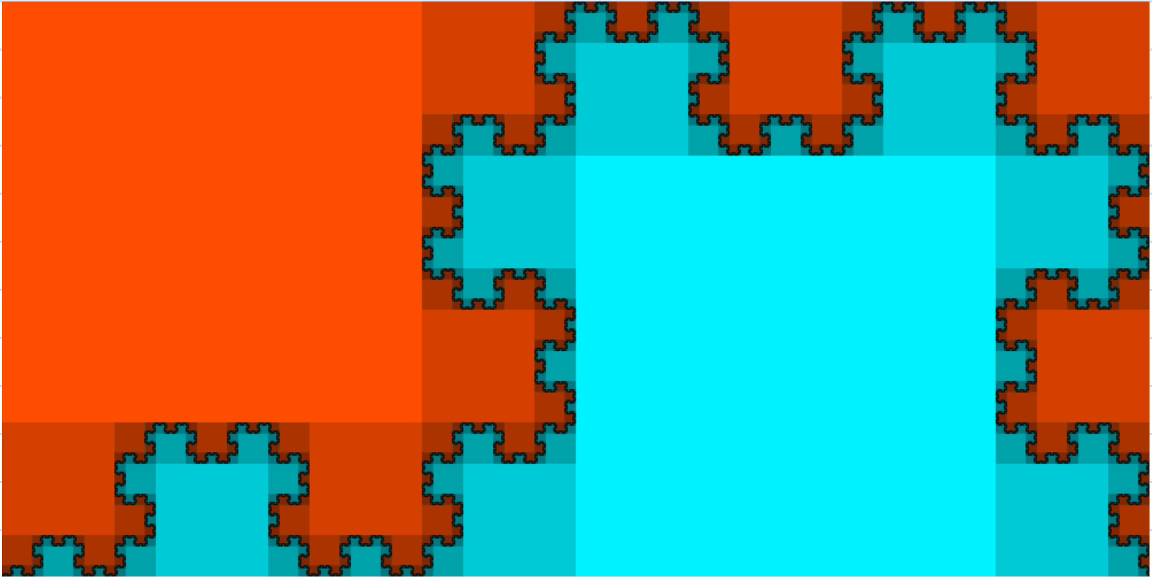
<u>Q1:</u> Is the action of $S_{\alpha,\beta}$ on the aperiodic set measurably conjugate to a rotation?

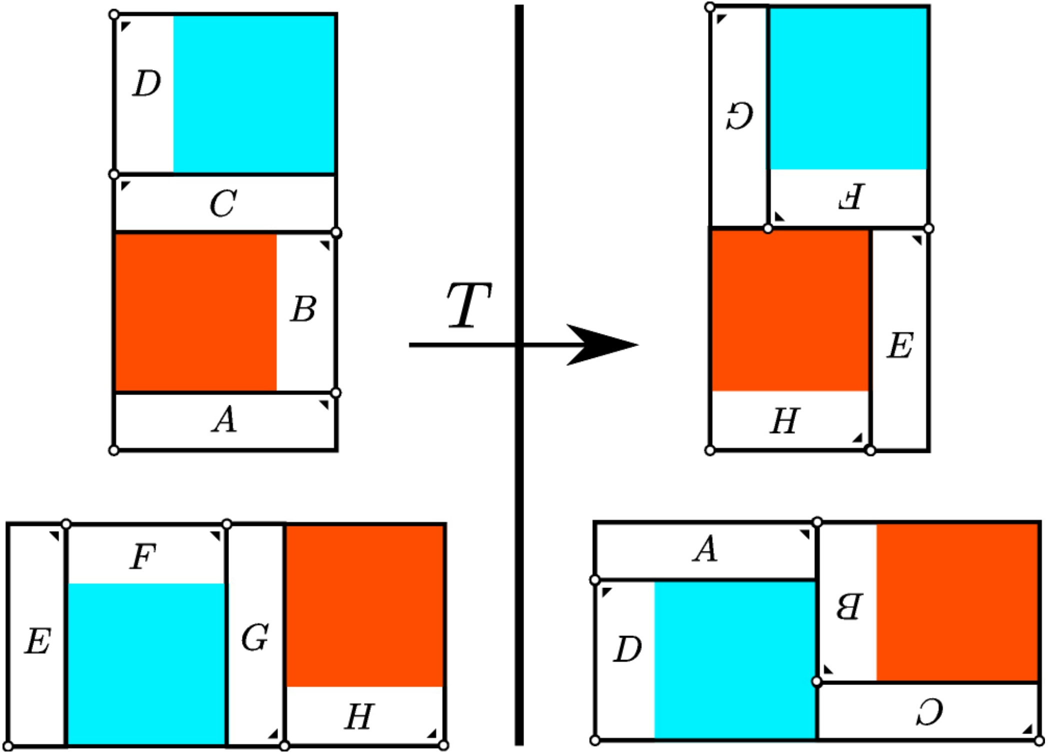
<u>Q2:</u> Is the action weak mixing in some cases? (That is, are there examples with no rotation appearing as a measurable factor?)

Part III

Substitutions and curves: An illustrative example.

Main Example: $\alpha = \beta = \frac{\sqrt{3}-1}{2}$

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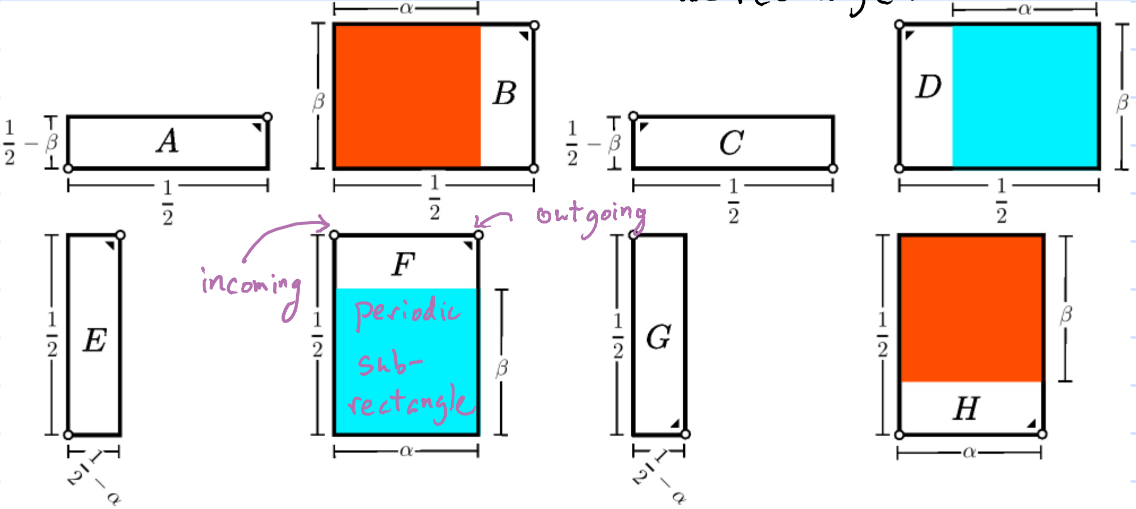
Decorated Rectangles:

We associate a decorated rectangle $R(L)$ to every $L \in \{A, \ldots, H\}$.

Decorations:

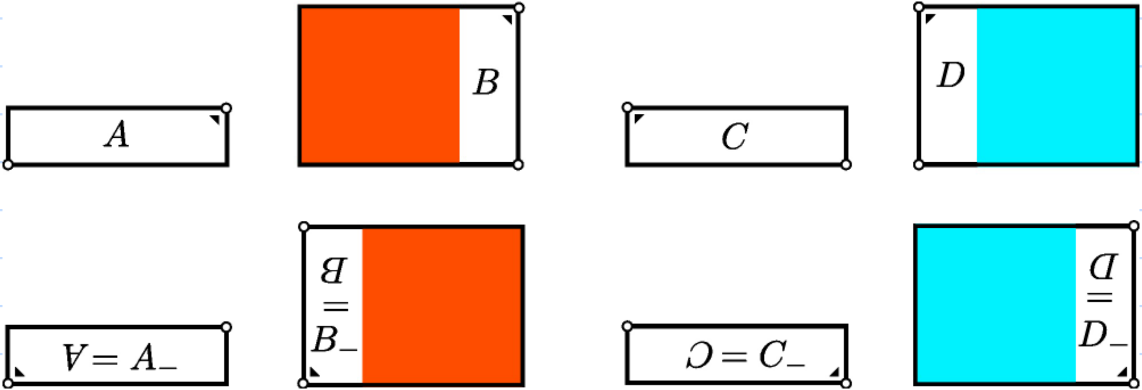
- Incoming and outgoing vertices.
- Sometimes, a "periodic subrectangle".



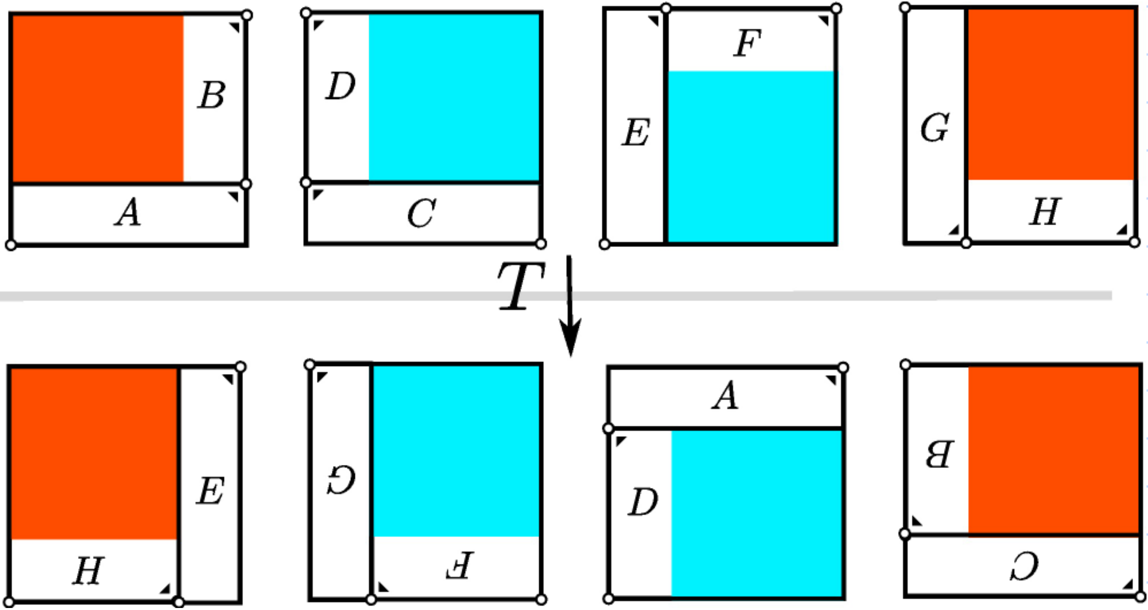
Flipped rectangles:

For each letter $L \in \{A, \dots, H\}$, we also define a flipped decorated rectangle $R(L_-)$.

We define the alphabet $\mathcal{L}_\pm = \{A, \dots, H\} \cup \{A_-, \dots, H_-\}$.
We define $\mathrm{neg} : \mathcal{L}_\pm \circlearrowleft$ to swap each L with L_-.



An alternate description of T:



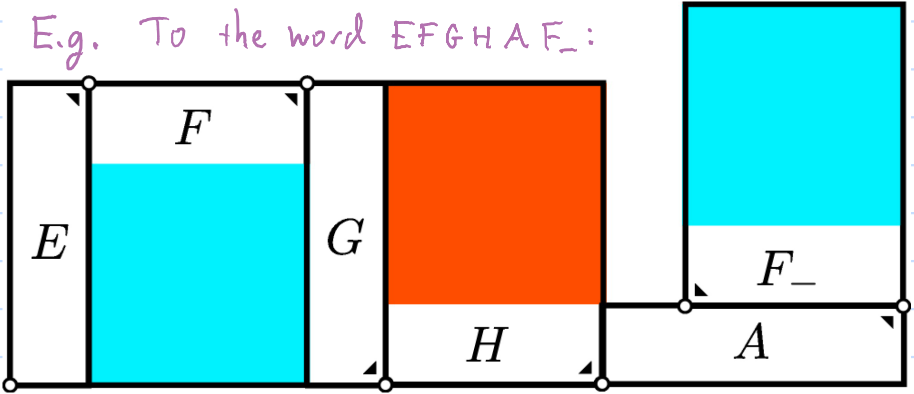
Chains of decorated rectangles:

To each word in the alphabet

$$\mathcal{L}_\pm = \{A,\ldots,H\} \cup \{A_-,\ldots,H_-\}$$

we associate a sequence of decorated rectangles.

E.g. To the word EFGHAF_:



A "magic" substitution:

The substitution Φ commutes with negation and is defined by:

$\Phi(A) = HEFGHDA$ $\qquad$ $\Phi(B) = BCDAB$

$\Phi(C) = CDH\_E\_F\_G\_H\_$ $\qquad$ $\Phi(D) = DABCD$

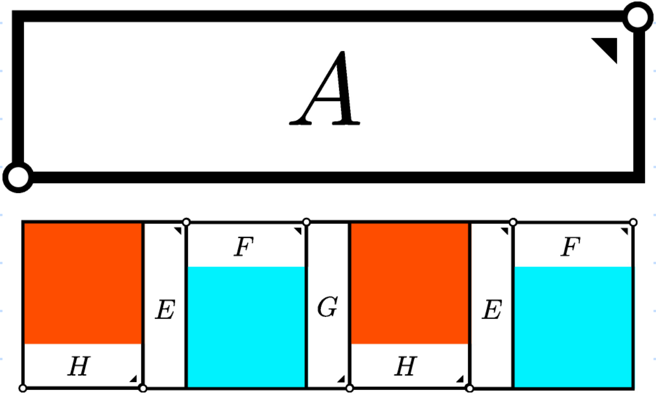
$\Phi(E) = DABCDHE$ $\qquad$ $\Phi(F) = FGHEF$

$\Phi(G) = GHD\_A\_B\_C\_$ $\qquad$ $\Phi(G) = HEFGH$

So, $\Phi(AB\_) = HEFGHDAB\_C\_D\_A\_B\_$.

Property 1:

For each $L \in \{A, \ldots, H\}$, the chain associated to $\Phi(L)$ scaled by $2-\sqrt{3}$ fills all of $R(L)$ except the periodic subrectangle and has the same incoming and outgoing vertex.



$\Phi(A) = HEFGHEF$

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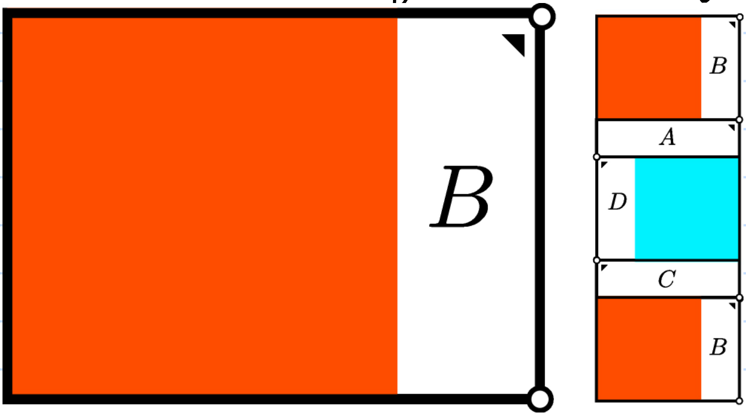
For each $L \in \{A, \dots, H\}$, the chain associated to $\Phi(L)$ scaled by $2-\sqrt{3}$ fills all of $R(L)$ except the periodic subrectangle and has the same incoming and outgoing vertex.

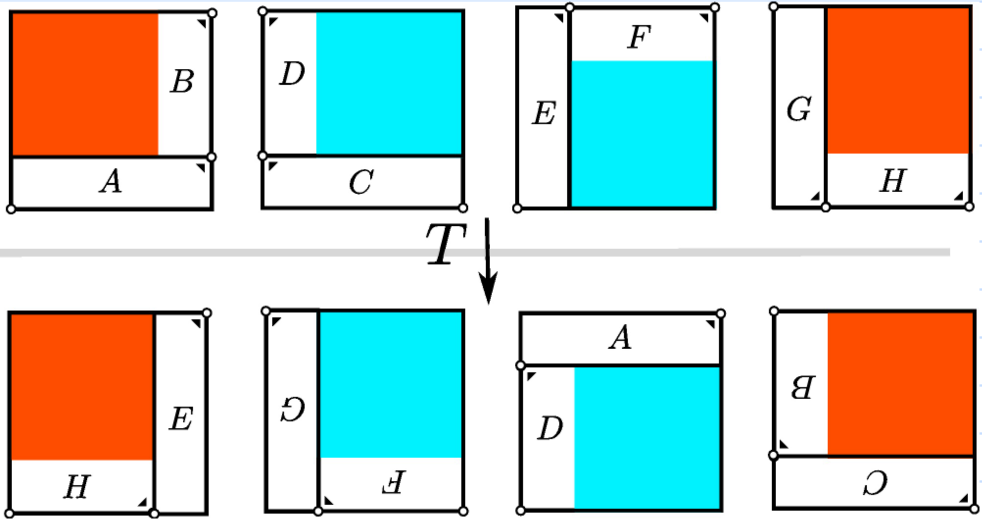


$\Phi(B) = BCDAB.$

Recall the alternate description of T:

Property 2 involves the collection of pairs

$AB \leftarrow HE$, $CD \leftarrow F_- G_-$, $EF \leftarrow DA$, $GH \leftarrow B_- C_-$



Property 2:

For every $w_1 \leftarrow w_2$,

$\Phi(w_2)$ can be obtained from $\Phi(w_1)$ by replacing one instance of w_1 in $\Phi(w_1)$ with w_2.

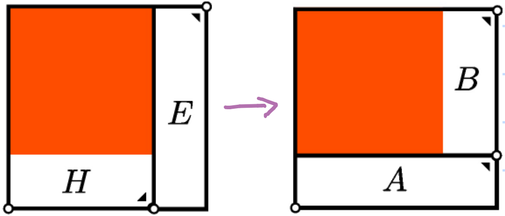
E.g. $\Phi(AB) = HEFGHDABCD\underline{AB}$

$\Phi(HE) = HEFGHDABCD\underline{HE}$

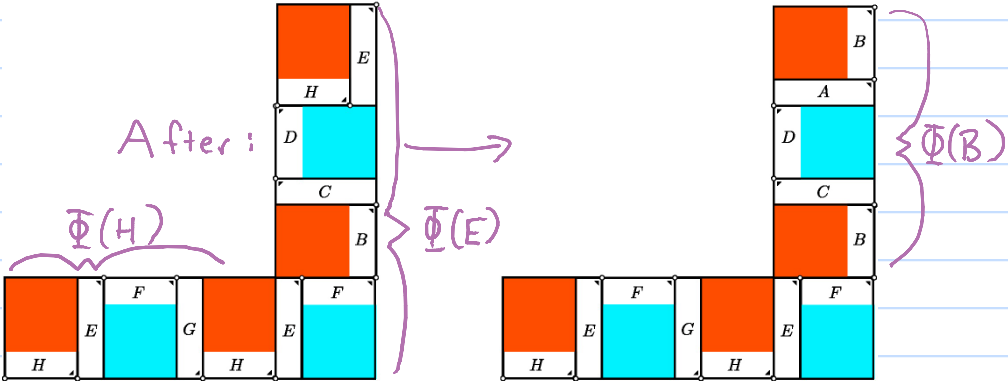
Observe: Powers of Φ also have this property.

Refining the dynamics:

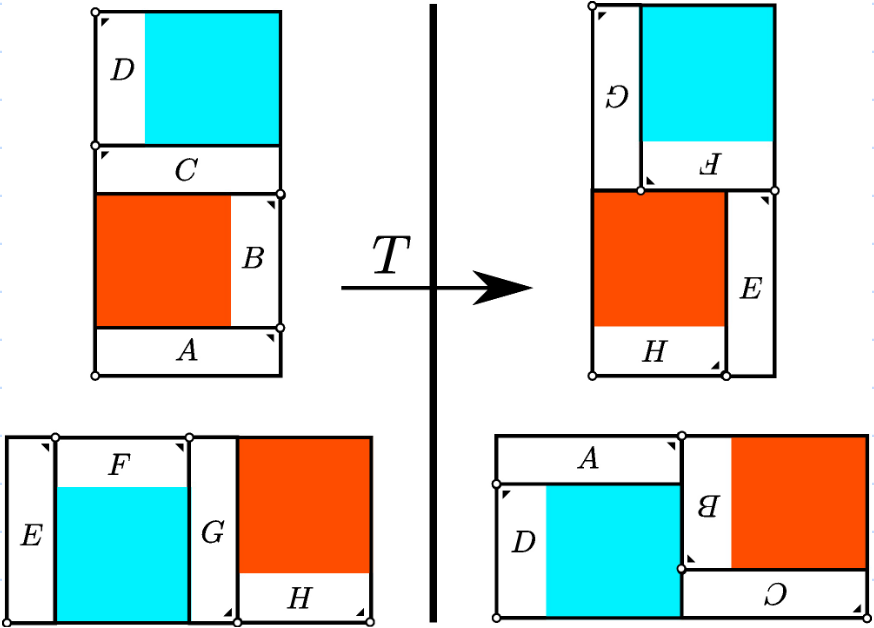
Before:



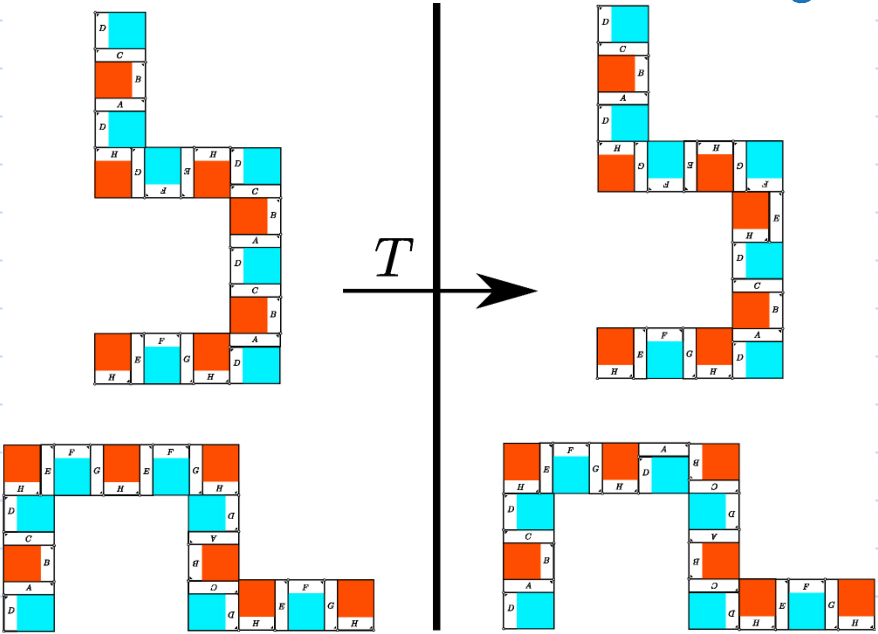
After:



Induced action on subrectangles.

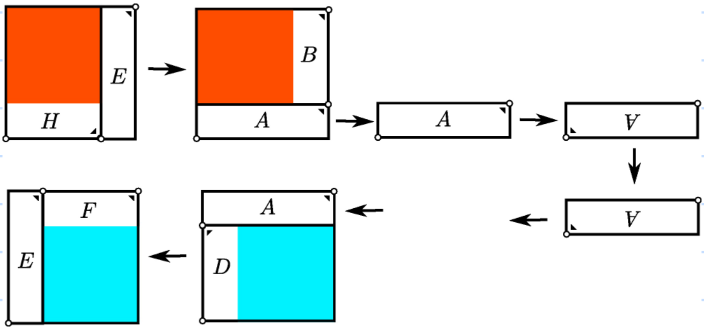


Induced action on subrectangles.



Possible partial T-orbits of a subrectangle.

① Birth, a finite orbit, and then death:



② Periodic motion. (Never happens.)

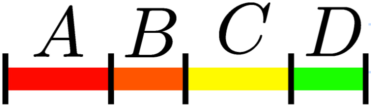
$$\left(AB \leftarrow HE,\quad CD \leftarrow F_- G_-,\quad EF \leftarrow DA,\quad GH \leftarrow B_- C_-\right)$$

Property 3:

The same substitution can be used to code an isometry of two circles.

$$\tau : \mathbb{R}/\mathbb{Z} \times \{1,2\} \circlearrowleft; \quad \tau(x,j) = \left(x + \tfrac{\sqrt{2}-1}{2} \pmod 1,\ 3-j\right).$$

$\mathbb{R}/\mathbb{Z} \times \{1\}$:

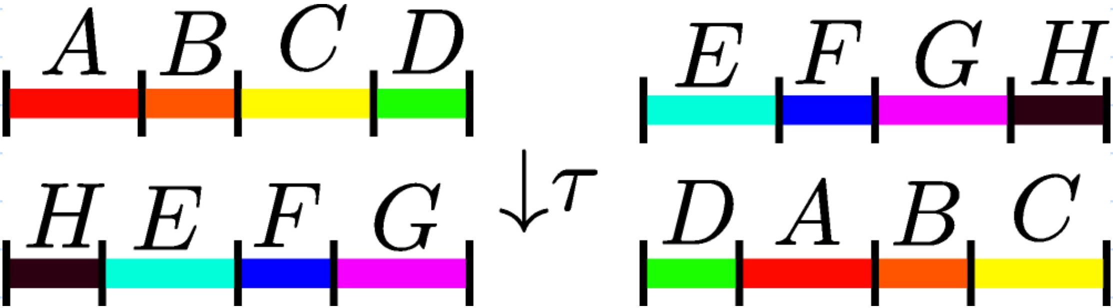


Intervals $A, C, E,$ & G have length $\frac{2-\sqrt{2}}{2}$.

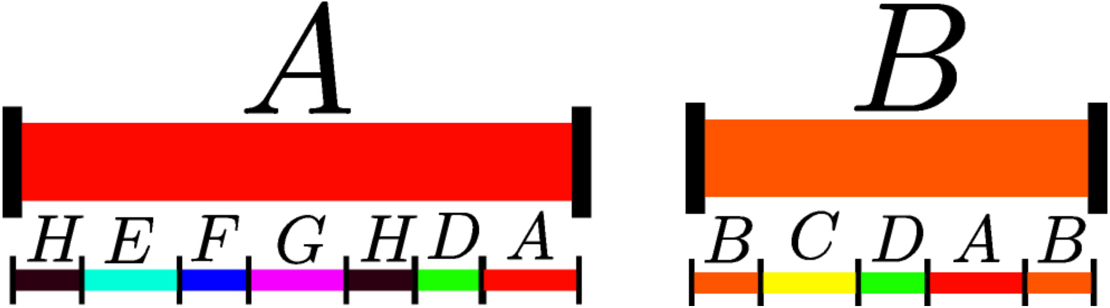
$\mathbb{R}/\mathbb{Z} \times \{2\}$:

Intervals $B, D, F,$ & H have length $\frac{\sqrt{2}-1}{2}$.

Action of the isometry:



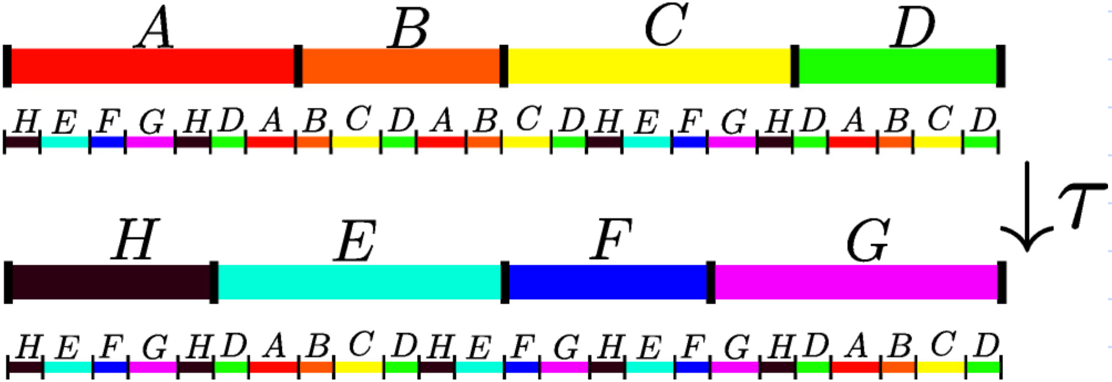
Interval Substitutions:



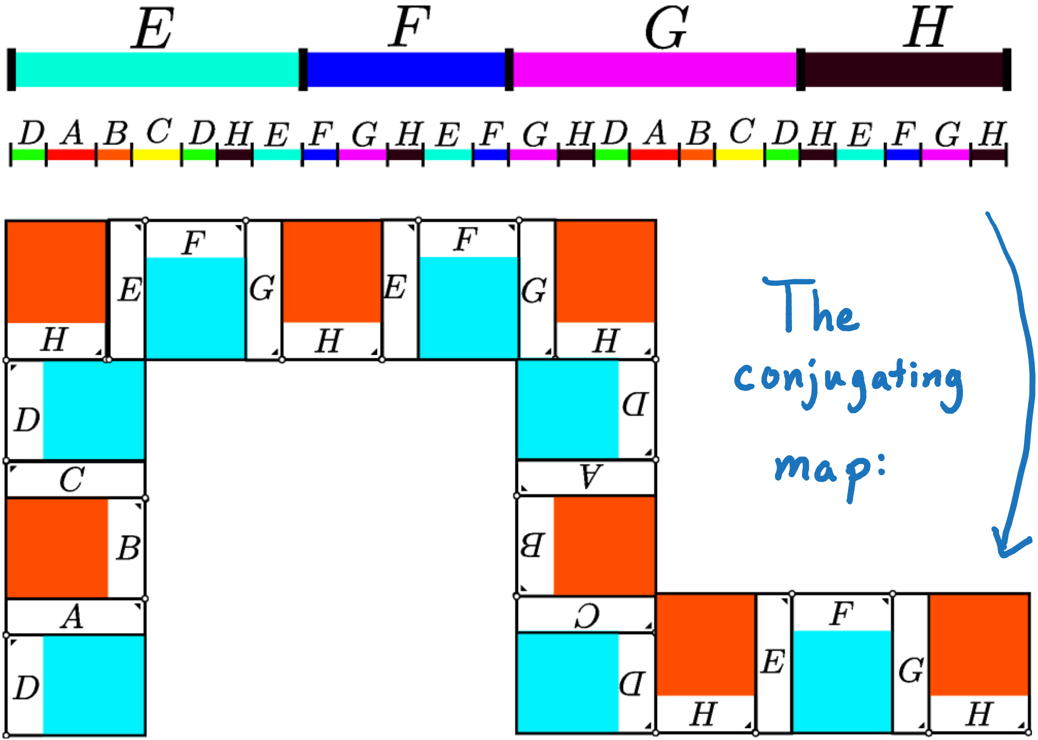
Let $\pi : \mathcal{L}_\pm \to \mathcal{L} = \{A, \dots, H\}$ be sign forgetting.

The interval associated to $\pi \circ \Phi(L)$ is $3+2\sqrt{2}$ times as long as the interval associated to $L \in \mathcal{L}$.

Refined Dynamics



Observe: No interval moves periodically, because τ^2 rotates each circle irrationally.


E
F
G
H
D A B C D H E F G H E F G H D A B C D H E F G H E F G H
H E F G H E F G H
D C B A D
D A B C D
H E F G H
The conjugating map: