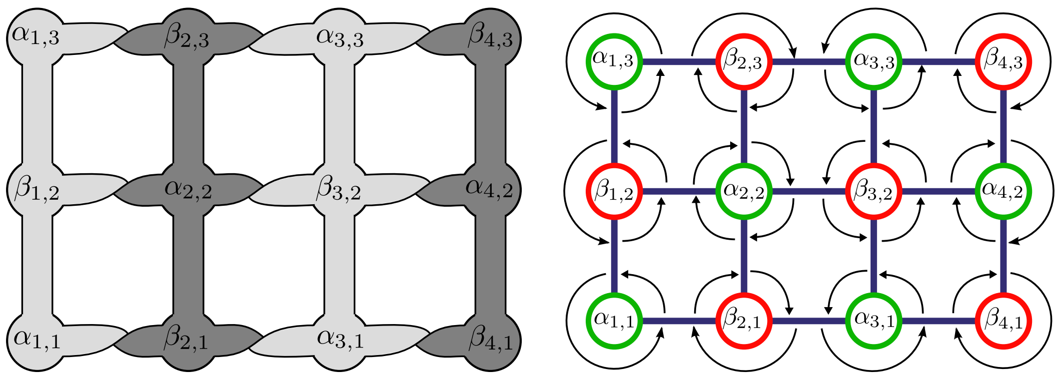
Grid graph description. We can think of $\mathbb{Z}^2$ as a graph by joining edges between points which differ by $(\pm 1, 0)$ or $(0, \pm 1)$. The grid graph $\mathcal{G}_{m,n}$ is subgraph with vertices in the set $\{1, \ldots, m-1\} \times \{1, \ldots, n-1\}$. We make this a ribbon graph using planar bands for vertical edges, and half-twisted bands for horizontal edges.

The left image below shows the ribbon graph $\mathcal{G}_{5,4}$. Let $\mathcal{E}$ be the edge set. Using the bipartite structure, we obtain two edge permutations $\mathfrak{e}, \mathfrak{n} : \mathcal{E} \to \mathcal{E}$ (see the image at right). The $\mathfrak{e}$ permutation uses the arrows around the α_* nodes, while the $\mathfrak{n}$ permutation uses the arrows around the β_* nodes.

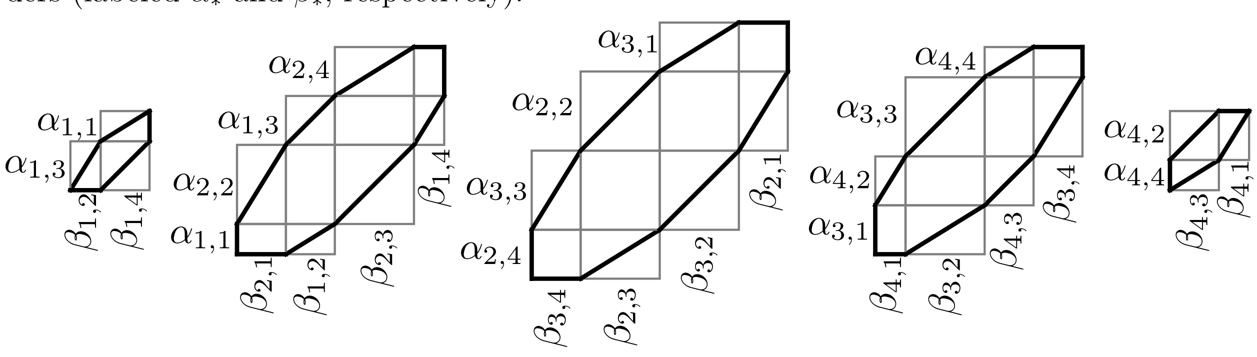


The positive eigenfunction is given by $w(i, j) = \sin(\frac{i\pi}{m}) \sin(\frac{i\pi}{n})$. This function has eigenvalue $\lambda = 2\cos(\frac{\pi}{m}) + 2\cos(\frac{\pi}{n})$. Thurston's construction shows that if we build a surface according to the bipartite ribbon graph $\mathcal{G}_{m,n}$ then the surface admits affine multitwists with derivatives $\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$. (When $\lambda > 2$, these two matrices do not generate a lattice.)

Thurston's construction of $(X_{m,n}, \omega_{m,n})$ from this data:

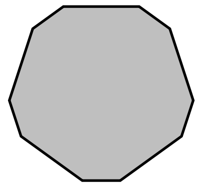
- For each $e = \overline{\alpha_{i,j}\beta k, l}$ construct $R_e = [0, w(\beta_{k,l})] \times [0, w(\alpha_{i,j})]$.
- Glue the right (east) side of R_e to the left side of $R_{\mathfrak{e}(e)}$ and the top (north) side to the bottom of $R_{\mathfrak{n}(e)}$.

The surface $(X_{5,5}, \omega_{5,5})$ is drawn below. Note that many rectangles appear twice, and should be identified. Other edge identifications are given by following horizontal and vertical cylinders (labeled α_* and β_*, respectively).



The dark edges above form a decomposition of $(X_{m,n}, \omega_{m,n})$ into affinely "semi-regular" $2n$-gons. They become "semi-regular" after applying the matrix $M = \begin{bmatrix} \csc(\frac{\pi}{n}) & \cot(\frac{\pi}{n}) \\ 0 & -1 \end{bmatrix}$ to the surface.

Semi-regular decomposition. A *semi-regular $2n$-gon* is a $2n$-gon with interior angles $\pi - \frac{\pi}{n}$, with even sides (resp. odd sides) all of the same length. We use $P_n(a, b)$ to denote the semi-regular $2n$-gon with even edges of length a and odd edges of length b. The following shows $P_5(1, 2)$.

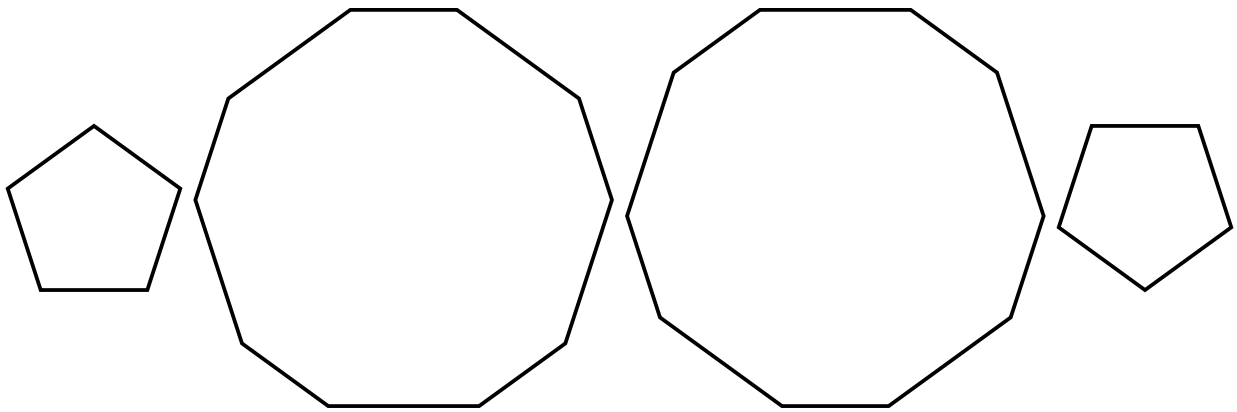


The surface $(Y_{m,n}, \eta_{m,n})$ is built from the semi-regular $2n$-gons:

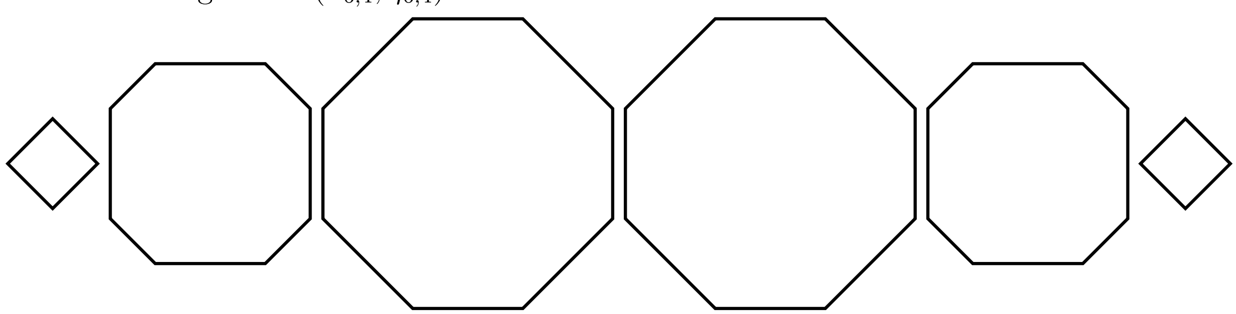
$$P_n(0, \sin \frac{\pi}{m}) \cup P_n(\sin \frac{\pi}{m}, \sin \frac{2\pi}{m}) \cup \ldots \cup P_n(\sin \frac{(m-2)\pi}{m}, \sin \frac{(m-1)\pi}{m}) \cup P_n(\sin \frac{(m-1)\pi}{m}, 0)$$

with edges of adjacent polygons of equal length identified by translation. (The polygons may need to be rotated to accomplish this.)

The following shows $(Y_{4,5}, \eta_{4,5})$:



The following shows $(Y_{6,4}, \eta_{6,4})$:



Note that $(Y_{4,5}, \eta_{4,5})$ admits a rotation of order $2n$ which switches the order of the polygons. This happens unless m and n are both even, explaining why when m and n are even, the Veech group is an index two subgroup of $\Delta^+(m, n, \infty)$. The fact that $M(X_{m,n}, \omega_{m,n}) = (Y_{m,n}, \eta_{m,n})$ implies that $M^{-1}RM$ is in $SL(X_{m,n}, \omega_{m,n})$ where R is a rotation by $\frac{\pi}{n}$ when m or n is odd and a rotation by $\frac{2\pi}{n}$ when m and n are even. This element together with the parabolics obtained from Thurston's construction generate the orientation preserving part of the Veech group $SL(X_{m,n}, \omega_{m,n})$.