Infinite genus surfaces admitting two affine multi-twists
or
The invariant measures of some infinite interval exchange maps

Pat Hooper
City College of New York

CCSU Teichmuller Theory

April 2010

Talk Outline:

1. Translation surfaces (finite and infinite)
2. Deterministic random walks
3. Masur's criterion for unique ergodicity of the straight-line flow
4. Thurston's construction of pseudo-Anosov automorphisms of surfaces
5. Results on the infinite version of Thurston's construction
6. Invariant measures for some deterministic random walks

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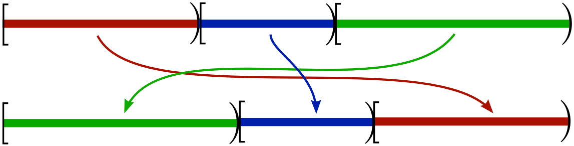
What is an interval exchange map?

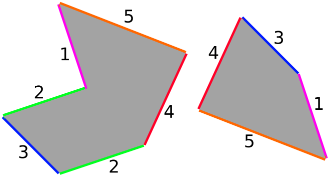
Def. An **interval exchange map** is a bijective piecewise orientation preserving isometry of an interval with finitely many discontinuities.



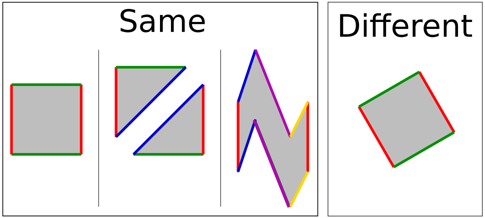
Dynamicists are interested in IEMs because they are the simplest examples of “low complexity” dynamics. (Nearby points tend to stay nearby for a very long time.) In that sense, they are natural generalizations of irrational rotations.

What is a translation surface?

Def 1. A **translation surface** is a surface built from a finite collection of polygons in $\mathbb{R}^2$ with edges identified in pairs by translations.
A **half translation surface** also allows gluing by rotations by π.



Def 2. A **translation surface** is a Riemann surface X equipped with a holomorphic one-form ω. We denote it (X, ω).
A **half translation surface** is a pair (X, ϕ) where ϕ is a quadratic differential.



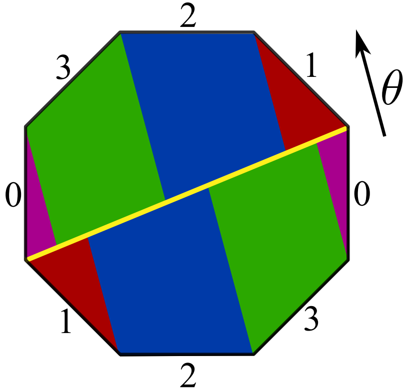
IEMs and Translation surfaces

Let S be a translation surface and $\mathbf{u}_\theta \in \mathbb{R}^2$ be the unit vector pointing in direction θ.

The **straight-line flow in direction θ** is the flow $F_t^\theta : (X, \omega) \to (X, \omega)$ defined in local coordinates by $F_t^\theta(x, y) = (x, y) + t\mathbf{u}_\theta$.

Fact. Let $\sigma \subset (X, \omega)$ be a line segment pointed in a direction different than θ. Then, the return map $T : \sigma \to \sigma$ is an interval exchange map.

Conversely, every interval exchange map can obtained in this way. (This is Veech's zippered rectangle construction.)



What is an infinite translation surface?

An **infinite translation surface** is a surface built from an infinite collection of polygons in $\mathbb{R}^2$ with edges identified in pairs by translations.

Why should I care?

- The Zemlyakov-Katok unfolding construction relates billiards in polygons with irrational angles to an ITS.
- Aaronson-Nakada-Sarig-Solmyak studied a family of infinite interval exchange maps called skew rotations. Hubert-Weiss reinterpreted their results using an ITS.
- ITSs appear as limits of finite translation surfaces.
- In addition to people interested in irrational billiards, lots of people are now studying them: Bowman, Hubert, Lelievre, Przytycki, Ralston, Schmithüsen, Troubetzkoy, Valdez, Weiss.
- It is becoming apparent that there will be generalizations of results about finite translation surfaces that will apply to ITSs.

Outline

What is a deterministic random walk (on a group)?

Random walks:

Let $\mathbb{G}$ be a group, and let $\gamma_1, \ldots, \gamma_6 \in \mathbb{G}$, where 6 denotes some positive integer larger than one.

Definition (Random walk)

A **random walk** on the group $\mathbb{G}$ is when $g \in \mathbb{G}$ moves to $\gamma_i g$ if the dice rolls i.

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Deterministic random walks:

Let $T : X \to X$ be a dynamical system, and let $\psi : X \to \{1, \ldots, 6\}$.
The pair (T, ψ) constitute a "random number generator."

Definition

A **skew product** of $\mathbb{G}$ and T a dynamical system $\widetilde{T} : \mathbb{G} \times X \to \mathbb{G} \times X$ given by $(g, x) \mapsto \big(\gamma_{\psi(x)} g, T(x)\big)$.

The projection of an orbit of $\widetilde{T}$ to $\mathbb{G}$ is a **deterministic random walk**.

Example: Skew rotations

- Suppose $\mathbb{G}$ is generated by $\gamma_1, \ldots, \gamma_k$ and
- Let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an irrational rotation

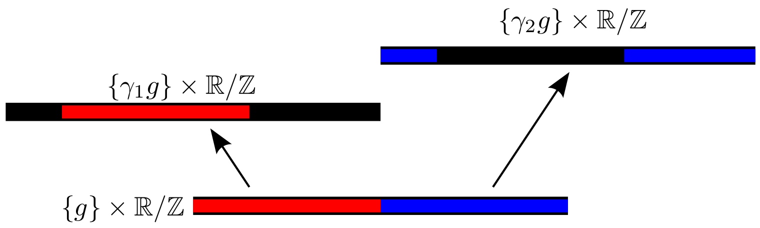
$$x \mapsto x + \alpha \pmod 1 \quad \text{with } \alpha \in \mathbb{R} \smallsetminus \mathbb{Q}.$$

- Set $\psi : \mathbb{R}/\mathbb{Z} \to \{1, \ldots, k\}$ so that $\psi([\frac{i-1}{k}, \frac{i}{k})) = i$ for $i = 1, \ldots k$.

Definition

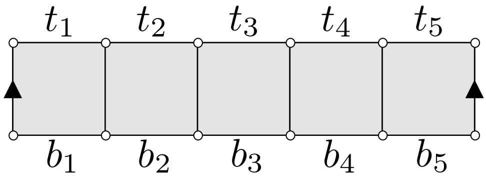
The skew product on $\mathbb{G} \times \mathbb{R}/\mathbb{Z}$ is called a **skew rotation**:

$$\widetilde{T}(g, x) \mapsto \big(\gamma_{\psi(x)} g, x + \alpha \pmod 1\big).$$



Skew rotations and translation surfaces

Let C be the cylinder $\mathbb{R}/\mathbb{Z} \times [0, \frac{1}{k}]$, with boundary edges labelled as below.



Let S be the surface $C \times \mathbb{G}$ with edge t_i of $\{g\} \times C$ identified with edge b_i of $\{\gamma_i g\} \times C$.

Recall the map $\widetilde{T}(g, x) = \big(\gamma_i g, x + \alpha \pmod 1\big)$ when $x \in [\frac{i-1}{k}, \frac{i}{k})$. This map arises from the return map of the straight-line flow in direction $(\alpha, 1/k)$ to the horizontal edges of S.

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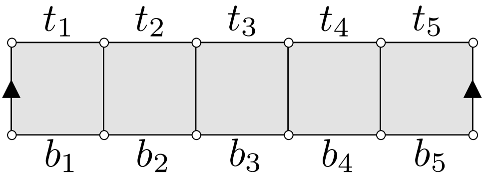

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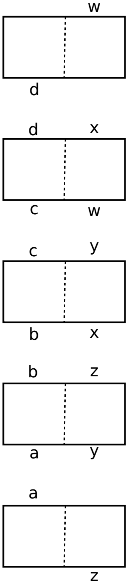
$\mathbb{G} = \mathbb{Z}$, $\gamma_1 = 1$ and $\gamma_2 = -1$:



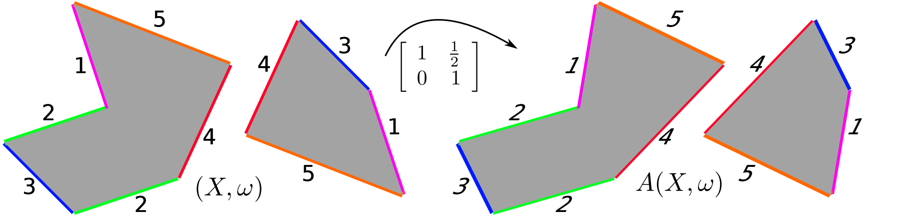
Outline

Moduli space and the $SL(2,\mathbb{R})$-action

Irrelevantly, the space of all quadratic differentials forms the cotangent bundle to the moduli space of all Riemann surfaces $\mathcal{M}_g$.

Let QD_g denote the collection pairs (X,ϕ) where ϕ has area one.

The group $SL(2,\mathbb{R})$ acts on QD_g. This action is induced by the linear action on polygons in $\mathbb{R}^2$.



Given $(X,\phi) \in QD_g$, the map $\Phi_{(X,\phi)} : SL(2,\mathbb{R}) \to QD_g$ sending $A \in SL(2,\mathbb{R})$ to $A(X,\phi)$ is an immersion. (Veech)

The group $\Gamma(X,\phi) = \{A \in PSL(2,\mathbb{R}) \ : \ A(X,\phi) = (X,\phi)\}$ is the **Veech group of** (X,ϕ). $\Phi_{(X,\phi)}$ induces an embedding of $SL(2,\mathbb{R})/\Gamma(X,\phi)$ into QD_g.

Masur's recurrence criterion for unique ergodicity

Let θ be a direction, let $R_\theta \in SL(2,\mathbb{R})$ be the rotation by θ, $E_t = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{bmatrix}$.

Set $A_{\theta,t} = R_\theta \circ E_t \circ R_\theta^{-1}$.

A path $\gamma : [0,\infty) \to X$ in a topological space X is said to **recur** if there is a $K \subset X$ compact with $\sup\ \gamma^{-1}(K) = \infty$.

Theorem (Masur)

Suppose the straight-line flow in direction θ is **minimal** *(every infinite trajectory is dense).*
If the path $t \mapsto A_{\theta,t}(X,\phi)$ recurs in $\mathcal{M}_g$, then the straight-line flow in direction θ is **uniquely ergodic** *(admits only one invariant probability measure).*

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If $\Gamma \subset SL(2,\mathbb{R})$ is discrete, then the **horocyclic limit set** is

$$\begin{aligned} \Lambda_h\Gamma &= \left[\{\theta \ : \ t \mapsto A_{\theta,t} \text{ recurs in } SL(2,\mathbb{R})/\Gamma\}\right] \\ &= \left[\{\theta \ : \ \text{for all } \epsilon > 0 \text{ there exists } g \in \Gamma \text{ such that } \|g(\mathbf{u}_\theta)\| < \epsilon\}\right] \end{aligned}$$

Corollary

Given minimality, if $[\theta] \in \Lambda_h\Gamma(X,\phi)$ then the straight-line flow in direction θ is uniquely ergodic.

Infinite genus?

Let TS_∞ denote the space of infinite genus translation surfaces (with some hypothetical topology).

It is difficult to say what it means for $t \mapsto A_{\theta,t}(X,\omega)$ to recur when $(X,\omega) \in \mathcal{M}_\infty$.

Regardless of our definition of recur, $t \mapsto A_{\theta,t}(X,\omega)$ should still recur when $[\theta] \in \Lambda_h\Gamma(X,\omega)$, because the path recurs inside $SL(2,\mathbb{R})/\Gamma(X,\omega)$ which embeds in $\mathcal{M}_\infty$.

Motivational Question

What can be said about the straight-line flow in direction θ when $[\theta] \in \Lambda_h\Gamma(X,\omega)$?

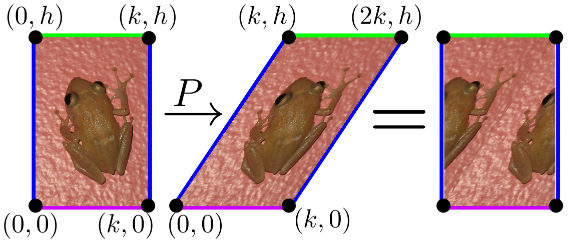
Outline

Affine Dehn twists

Let $C = (\mathbb{R}/k\mathbb{Z}) \times [0, h]$ be a Euclidean cylinder of circumference k and height h.

There is an automorphism $\phi : C \to C$ such that

- $D\phi(x) = \begin{bmatrix} 1 & \frac{k}{h} \\ 0 & 1 \end{bmatrix}$ for all $x \in C$.
- ϕ fixes every point in ∂C.



The constant $\frac{k}{h}$ is the **inverse modulus** of C.

Multi-twists

A **cylinder decomposition** of (X, ω) in direction θ is a direction decomposition into Euclidean cylinders with all circumferences of cylinders parallel to θ.

Fact: If λ is the inverse modulus of all cylinders C_i in a decomposition, then there exists an automorphism ϕ of (X, ω) such that

- $D\phi(x) = R_\theta \circ \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \circ R_\theta^{-1}$ for all $x \in (X, \omega)$.
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Thurston's Idea

If (X, ω) admits two multi-twists ϕ_1 and ϕ_2 in distinct directions then $\langle \phi_1, \phi_2 \rangle$ contains automorphisms with hyperbolic derivatives.

An automorphism of (X, ω) with hyperbolic derivatives induces **pseudo-Anosov** automorphism of the Riemann surface X.

Surfaces admitting two Multi-twists

Suppose (X, ω) admits two multi-twists in distinct directions.

By replacing (X, ω) with $A(X, \omega)$, we can assume that

- The multi-twists ϕ_h and ϕ_v are in horizontal and vertical directions.
- Each cylinder in either decomposition has inverse modulus λ.

With these assumptions, $\langle \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \rangle \subset \Gamma(X, \omega)$.

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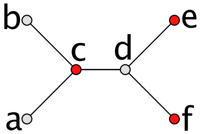
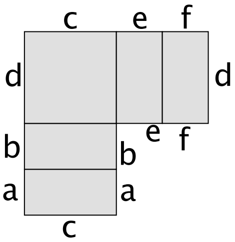
Construction of the cylinder intersection graph:

The vertices $\mathcal{V}$ are the horizontal and vertical cylinders.

Join an edge between a horizontal and a vertical cylinder for each intersection between them.





An example with $\lambda = 2$.

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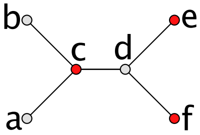

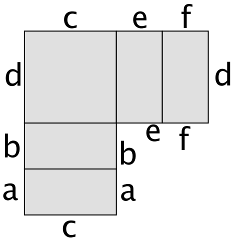

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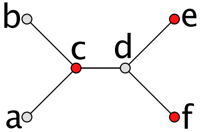

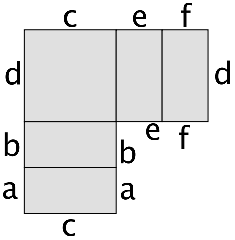

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We let $\mathbf{h} : \mathcal{V} \to \mathbb{R}$ be the function which assigns a cylinder to its height.

Then the circumference of a cylinder $x \in \mathcal{V}$ is given by $\mathbf{c}(x) = \sum_{y \sim x} \mathbf{h}(y)$.

Recall by assumption that $\mathbf{c}(x) = \lambda \mathbf{h}(x)$ for all $x \in \mathcal{V}$. That is, $\mathbf{A}(\mathbf{h}) = \lambda \mathbf{h}$ where $\mathbf{A} : \mathbb{R}^{\mathcal{V}} \to \mathbb{R}^{\mathcal{V}}$ is the adjacency operator defined by

$$\mathbf{A}(\mathbf{h})(x) = \sum_{y \sim x} \mathbf{h}(y) \quad \text{for all } x \in \mathcal{V}.$$

Thurston's construction

Observation (Thurston)

Let $\mathcal{G}$ be any finite connected bipartite graph, and let $\mathcal{V}$ be the vertex set.

By the Perron-Frobenius theorem, there is a positive function $\mathbf{h} : \mathcal{V} \to \mathbb{R}$ *satisfying* $\mathbf{A}(\mathbf{h}) = \lambda \mathbf{h}$ *for some* λ. *This* $\mathbf{h}$ *us unique up to scaling.*

There is a translation surface with horizontal and vertical cylinder decompositions, whose cylinder intersection graph is $\mathcal{G}$, *and whose cylinder heights are given by* $\mathbf{h}$.

All the cylinders have inverse modulus λ, *so* $\left\langle \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \right\rangle \subset \Gamma(X, \omega)$.

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Observation (McMullen)

$\mathcal{G}$ *can be endowed with a ribbon graph structure which records the how the cylinders are constructed from rectangles.*

The translation surface $S(\mathcal{G}, \mathbf{h})$ *described above is uniquely determined.*

Outline

Eigenfunctions of infinite graphs

For the rest of the talk,

- $\mathcal{G}$ will be a infinite connected bipartite ribbon graph with bounded valance.
- The vertex set $\mathcal{V}$ will be countably infinite.
- $\mathbf{A} : \mathbb{R}^{\mathcal{V}} \to \mathbb{R}^{\mathcal{V}}$ will be the adjacency operator.

For $\lambda > 0$, let $E(\lambda) = \{\text{positive } \mathbf{f} \in \mathbb{R}^{\mathcal{V}} ~:~ \mathbf{A}(\mathbf{f}) = \lambda \mathbf{f}\}$. Then $\{\mathbf{0}\} \cup E(\lambda)$ is a closed convex cone in the topology of pointwise convergence.

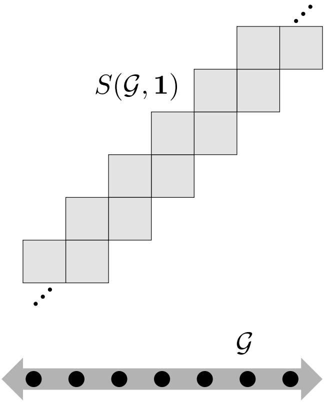
Fact. Let $r > 0$ be the **spectral radius** for the action of $\mathbf{A}$ on $\ell_2(\mathcal{V}) \subset \mathbb{C}^{\mathcal{V}}$. (That is, r is the supremum of $z > 0$ so that the action of $z\mathbf{I} - \mathbf{A}$ on $\ell_2(\mathcal{V})$ is non-invertible.) Then, $E(\lambda)$ is non-empty if and only if $\lambda \geq r$.

The spectral radius is bounded from above by the greatest valance of a vertex and from below by 2.

Def. We call $\mathbf{f} \in E(\lambda)$ **extremal** if whenever $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$ with $\mathbf{f}_1, \mathbf{f}_2 \in E(\lambda)$ we have $\mathbf{f}_1 = t\mathbf{f}$ and $\mathbf{f}_2 = (1-t)\mathbf{f}$ for some $t \in (0, 1)$.

The infinite version of Thurston's construction

Below: The staircase studied by Hubert and Weiss.



Choose $\mathbf{h} \in E(\lambda)$. Without modification, Thurston's construction produces an infinite genus surface $S(\mathcal{G}, \mathbf{h})$ so that

$$G_\lambda = \langle \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \rangle$$

lies in the Veech group $\Gamma S(\mathcal{G}, \mathbf{h})$.

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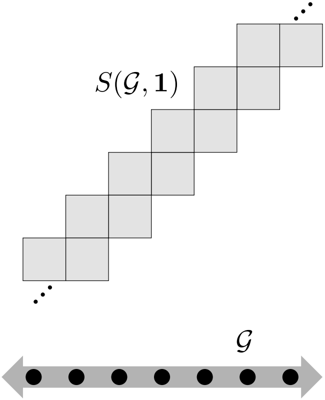

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Important definition:
A direction θ is called **λ-renormalizable** if

1. $[\theta] \in \Lambda_h G_\lambda$ (or equivalently for all $\epsilon > 0$ there is a $g \in G_\lambda$ so that $\|g(\mathbf{u}_\theta)\| < \epsilon$), and
2. θ is not an eigendirection of $h_\lambda = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix}$ or any conjugate of h_λ in G_λ.

Main results

Let $\mathcal{F}_\theta$ denote the foliation of a translation surface by orbits of the straight-line flow in direction θ.

Theorem (Orbit equivalence)

Suppose $\mathbf{h}_1, \mathbf{h}_2 \in \mathbb{R}^{\mathcal{V}}$ *are positive functions satisfying* $\mathbf{A}(\mathbf{h}_i) = \lambda_i \mathbf{h}_i$ *for* $i = 1, 2$. *Let* θ_1 *be a* λ_1*-renormalizable direction. Then there is a* λ_2*-renormalizable direction* $\theta_2 = \theta_2(\theta_1, \lambda_1, \lambda_2)$ *and a homeomorphism*

$$\psi : S(\mathcal{G}, \mathbf{h}_1) \to S(\mathcal{G}, \mathbf{h}_2)$$

which carries $\mathcal{F}_{\theta_1} \subset S(\mathcal{G}, \mathbf{h}_1)$ *to* $\mathcal{F}_{\theta_2} \subset S(\mathcal{G}, \mathbf{h}_2)$.

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Let $\mathcal{M}_{\theta_1}$ denote the locally finite transverse measures to the foliation $\mathcal{F}_{\theta_1} \subset S(\mathcal{G}, \mathbf{h}_1)$.

Theorem (Ergodic measures)

Suppose in addition that $\mathcal{G}$ *has no vertices of valance one.*
Then the collection of ergodic measures in $\mathcal{M}_{\theta_1}$ *consist exactly of the pullbacks under* ψ *of the Lebesgue transverse measure to* $\mathcal{F}_{\theta_2}$ *on* $S(\mathcal{G}, \mathbf{h}_2)$*, where* $\mathbf{h}_2$ *varies over the extremal positive eigenfunctions of* $\mathbf{A}$.

Two rough ideas

1. Measures $\rightarrow$ cohomology:

Let $\mathcal{M}_\theta$ denote the locally finite transverse measures to $\mathcal{F}_\theta$.

We can make a $\mu \in \mathcal{M}_\theta$ into a signed measure $\mu_\pm$ by keeping track of the algebraic sign with which a transversal crosses $\mathcal{F}_\theta$.

Let $V \subset S(\mathcal{G}, \mathbf{h})$ denote the vertices of rectangles. Given $\mu \in \mathcal{M}_\theta$ we obtain a cohomology class $\Psi_\theta(\mu) \in H^1(S, V, \mathbb{R})$ by restricting $\mu_\pm$ to $H_1(S, V, \mathbb{R})$.

Lemma: If the straight-line flow in direction θ is Poincaré recurrent and has no periodic trajectories, then Ψ_θ is injective.

Two rough ideas

1. Measures $\to$ cohomology: $\Psi_\theta : \mathcal{M}_\theta \to H^1(S, V, \mathbb{R})$ is injective.

2. Renormalization:

Call $m \in H^1(S, V, \mathbb{R})$ a θ-survivor if for all horizontal and vertical edges of rectangles σ, the sign of $m(\llbracket\sigma\rrbracket)$ could have come from a transverse measure to $\mathcal{F}_\theta$. (This only depends on the quadrant in $\mathbb{R}^2$ containing θ.)

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Since θ is λ-renormalizable, there is a natural sequence

$$A_n \in G_\lambda = \langle \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \rangle$$

so that $A_n(\mathbf{u}_\theta)$ tends to zero monotonically. There are associated automorphisms $\phi_n : S(\mathcal{G}, \mathbf{h}) \to S(\mathcal{G}, \mathbf{h})$ with derivative A_n.

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If $m \in \Psi_\theta(\mathcal{M}_\theta)$, then $m \circ \phi_n^{-1}$ is a $A_n(\theta)$-survivor for all n.

Theorem:
$\Psi_\theta(\mathcal{M}_\theta) = \{m \in H^1(S, V, \mathbb{R}) \ : \ m \circ \phi_n^{-1} \text{ is a } A_n(\theta)\text{-survivor for all } n.\}$.

Uniquely ergodic examples

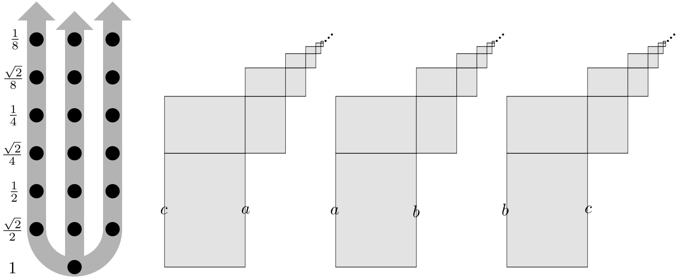
The area of the surface $S(\mathcal{G}, \mathbf{h})$ is given by $\frac{\lambda}{2}\|\mathbf{h}\|_2$. In particular, Lebesgue measure is a finite measure if and only if $\mathbf{h} \in \ell_2(\mathcal{V})$.

Theorem (Vere-Jones)

There is at most one ray of positive eigenfunctions of $\mathbf{A}$ in $\ell^2(\mathcal{V})$.

Corollary

Suppose $\mathbf{h} \in \ell^2(\mathcal{V})$ is a positive eigenfunction of $\mathbf{A}$. Then the straight-line flow on $S(\mathcal{G}, \mathbf{h})$ in any λ-renormalizable direction is uniquely ergodic.



Outline

Recall: Skew rotations

- Suppose $\mathbb{G}$ is generated by $\gamma_1, \ldots, \gamma_k$ and
- Let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an irrational rotation

$$x \mapsto x + \alpha \pmod 1 \quad \text{with } \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

- Set $\psi : \mathbb{R}/\mathbb{Z} \to \{1, \ldots, k\}$ so that $\psi\big([\frac{i-1}{k}, \frac{i}{k})\big) = i$ for $i = 1, \ldots k$.

Definition

The skew product on $\mathbb{G} \times \mathbb{R}/\mathbb{Z}$ is called a **skew rotation**:

$$\widetilde{T}(g, x) \mapsto \big(\gamma_{\psi(x)} g, x + \alpha \pmod 1\big).$$

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Example Invariant Measures:
Let $\chi : \mathbb{G} \to \mathbb{R}_\times$ be a group homomorphism to the multiplicative group of positive reals. Let μ be a measure on $X = \mathbb{R}/\mathbb{Z}$ so that $\frac{d\mu \circ T}{d\mu} = \chi \circ \psi$ μ-a.e.

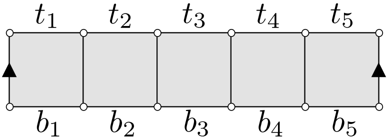
Let $\pi_g : X \to X \times \mathbb{G}$ be the map $x \mapsto (x, g)$. The measure $\widetilde{\mu}$ on $X \times \mathbb{G}$ so that $\widetilde{\mu} \circ \pi_g = \frac{1}{\chi(g)} \mu$ is called a **χ-Maharam measure**.

The measure $\widetilde{\mu}$ is $\widetilde{T}$ invariant, because of cancellation of χ factors.

Connecting skew rotations to Thurston's construction

We now assume $\gamma_k \gamma_{k-1} \ldots \gamma_1 = e$.

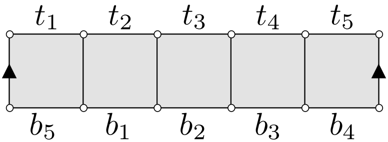
Let C be the cylinder $\mathbb{R}/\mathbb{Z} \times [0, \frac{1}{k}]$, with boundary edges as below.



Let S_0 be the surface $C \times \mathbb{G}$ with edge t_i of $C \times \{g\}$ identified with edge b_i of $C \times \{\gamma_i g\}$.

Recall the map $\widetilde{T}(x, g) = (x + \alpha, \gamma_i g)$ when $x \in [\frac{i-1}{k}, \frac{i}{k})$. This map arises from the return map of the flow in direction $(\alpha, 1/k)$ to the horizontal edges of S_0.

Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Then $A(S_0)$ is of the form $S(\mathcal{G}, \frac{1}{k}\mathbf{1})$ as $\gamma_k \gamma_{k-1} \ldots \gamma_1 = e$.



Corollaries for skew rotations

Recall, $\widetilde{T}(g, x) = (\gamma_i g, x + \alpha)$ when $x \in [\frac{i-1}{k}, \frac{i}{k})$ and $\gamma_k \gamma_{k-1} \ldots \gamma_1 = e$.

For the remainder assume $A(\alpha, 1/k) = (\alpha - 1/k, 1/k)$ is k-renormalizable.

Corollary

The locally finite ergodic measures of $\widetilde{T}$ are in bijective correspondence with the extremal positive eigenfunctions for the adjacency operator on the Cayley graph with generators $\gamma_1, \ldots, \gamma_k$.

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Corollary

For all homomorphisms $\chi : \mathbb{G} \to \mathbb{R}_\times$, the associated Maharam measure exists and is unique.

Nilpotent groups

Recall, $\widetilde{T}(x,g) = (x+\alpha, \gamma_i g)$ when $x \in [\frac{i-1}{k}, \frac{i}{k})$ and $\gamma_k \gamma_{k-1} \dots \gamma_1 = e$.
Assume $(\alpha - 1/k, 1/k)$ is k-renormalizable.

Theorem (Margulis, 1966)

If $\mathbb{G}$ is nilpotent, the extremal positive eigenfunctions are the group homomorphisms $\mathbb{G} \to \mathbb{R}_{\times}$.

Corollary

If $\mathbb{G}$ is nilpotent, the locally finite ergodic measures are precisely the Maharam measures.

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Conjecture

The condition that $(\alpha - 1/k, 1/k)$ is k-renormalizable can be replaced with α irrational.

Remark J-P Conze has results to this effect for skew products of rotations with nilpotent Lie groups when $\gamma_1, \dots, \gamma_k$ generate a dense subgroup.

Non-abelian free groups

Recall, $\widetilde{T}(x,g) = (x+\alpha, \gamma_i g)$ when $x \in [\frac{i-1}{k}, \frac{i}{k})$ and $\gamma_k \gamma_{k-1} \ldots \gamma_1 = e$.
Assume $(\alpha - 1/n, 1/n)$ is k-renormalizable.

Theorem (Picardello-Woess, 1988)

Suppose $\mathbb{G}$ is the free group generated by $\gamma_1, \ldots, \gamma_{k-1}$ for $n > 2$. The space of extremal positive eigenfunctions of the Cayley graph is isomorphic to the product of the Gromov boundary $\partial\mathbb{G}$ and the ray $[0, \infty)$.

Remark: Ancona has a result which is nearly this strong when $\mathbb{G}$ is Gromov hyperbolic.

Corollary

Again suppose $\mathbb{G}$ is free as above. Then no Maharam measure is ergodic for $\widetilde{T}$. Instead the space of locally finite ergodic invariant measures is isomorphic to $\partial\mathbb{G} \times [0, \infty)$.

Remark: Unlike the nilpotent case, it is easy to see in the free group case that there are positive measures of directions which are totally dissipative. In these directions the locally finite ergodic invariant measures are all atomic.