# Some irrational polygons have many periodic billiard paths

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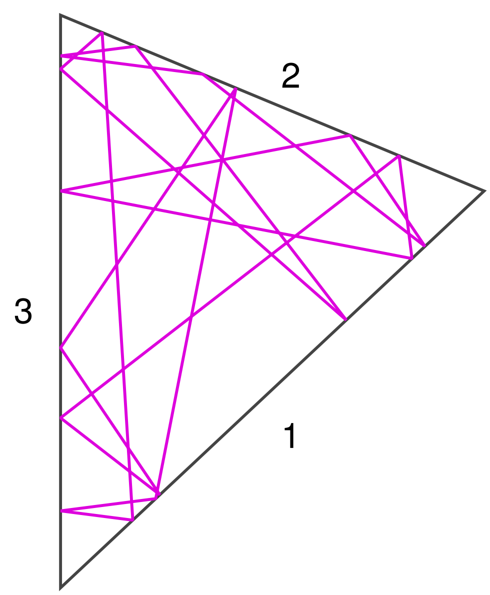
## Billiard trajectories in polygons

A **billiard path** in a polygon $P$ is a unit speed path which is

- geodesic in the interior,
- and bounces off $\partial P$ according to the laws of optics.

(The angle of incidence must equal the angle of reflection.)

Billiard trajectories are not defined through vertices.



## Periodic billiard paths in polygons

A **periodic billiard path** is a billiard path which returns to its starting point traveling in the same direction.

**Open Question**

*Does every polygon have a periodic billiard path?*

This question is open even for triangles.

We will survey some progress on this question.

A **rational polygon** is a polygon whose angles are all rational multiples of $\pi$.

**Theorem (Masur, 1986)**

*Every rational polygon has a periodic billiard path. (Moreover, there is a periodic billiard path in a dense set of directions.)*

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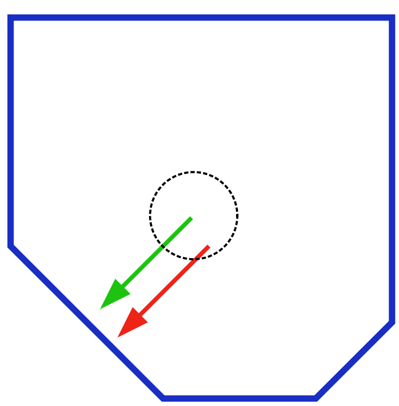
**Existence proof based on an observation of Schwartz:**
Rational polygons can be distinguished by the property that any billiard trajectory can only move in finitely many directions.

Consider the possible directions that can be reached from a trajectory aimed straight at a favorite edge.



The space of all vectors pointing in these directions has a finite invariant measure.

Apply Poincaré recurrence to an open set of trajectories aimed straight at a wall. **Q.E.D.**

## Periodic billiard paths in irrational polygons

**Theorem (Fagnano, 1755)**

*Acute triangles have a periodic billiard path.*

**Theorem (Schwartz, 2005)**

*Triangles whose largest angle is less than* 100 *degrees have periodic billiard paths.*

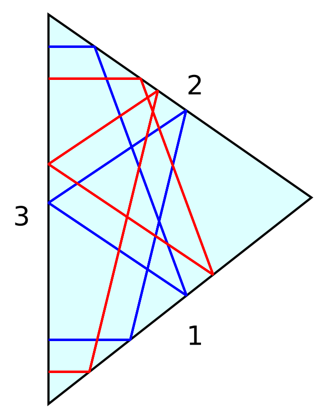
**Theorem (H.-Schwartz, 2006)**

*Nearly isosceles triangles have periodic billiard paths.*

*(If $T_n$ is a sequence of triangle converging to an isosceles triangle, then there is an N such that $T_n$ has a periodic billiard path for $N > n$.)*

## Counting periodic billiard paths

Let $N(P,t)$ denote the number of (combinatorially distinct) periodic billiard paths in the polygon $P$ of length less than $t$.




**Theorem (Masur, 1986 & 1990)**

*Let $P$ be a rational polygon.*

$$0 < \liminf_{t\to\infty} \frac{N(P,t)}{t^2} \leq \limsup_{t\to\infty} \frac{N(P,t)}{t^2} < \infty.$$

**Theorem (Veech, 1989)**

*Let $T$ be an isosceles triangle with angles $(\frac{\pi}{n}, \frac{\pi}{n}, \frac{(n-2)\pi}{n})$. Then* $\lim_{t\to\infty} N(T,t)$ *converges. (Veech gives the limit.)*

## Counting periodic billiard paths in irrational polygons

What can we hope to say about irrational polygons?

**Question**

*How fast can $N(P, t)$ grow in an irrational polygon $P$?*

**Theorem (H., 2007)**

*For all $n \geq 3$ and all $k > 0$, there is an $n$-gon $P$ such that*

$$\liminf_{t \to \infty} \frac{N(P, t)}{t \log^k t} > 0.$$

## The theorem holds uniformly on an open set

Let $P$ be an $n$-gon. Mark the edges of $P$ by the numbers $1 \ldots n$.

The **orbit type** of a periodic billiard path is the bi-infinite sequence of edges the billiard path hits.

**Remark:** The period of an orbit type of a periodic billiard path in a fixed polygon $P$ is comparable to the path's Euclidean length.

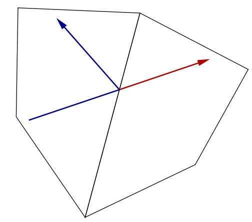
Let $U$ be set of marked $n$-gons. Let $N(U, t)$ denote the number of orbit types $\mathcal{O}$ of period less than $t$ such that each $P \in U$ has a periodic billiard path with orbit type $\mathcal{O}$.

**Main Theorem (H., 2007)**

*For all $n \geq 3$ and all $k > 0$, there is an open set $U$ of $n$-gons such that*

$$\liminf_{t \to \infty} \frac{N(U, t)}{t \log^k t} > 0.$$

## Translation surfaces



Consider the simplifying observation to the right.

Let $P$ be a polygon and $G \subset O(2)$ be the subgroup of the orthogonal group generated by the orthogonal parts of reflections in the sides of $P$.

The **translation surface associated to $P$**, denoted $S(P)$, is $\bigsqcup_{g \in G} g(P)/\sim$. Here $\sim$ identifies edges of the polygons in pairs.

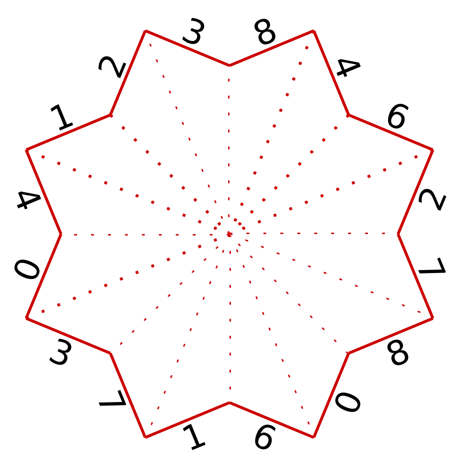
If $e$ is an edge of $P$, then $g_1(e) \subset g_1(P)$ is identified to $g_2(e) \subset g_2(P)$ by translation if the two edges are parallel.

Group theoretically, $g_1 \circ g_2^{-1}$ must be reflection in the side $e$.

In general, A **translation surface** is a union of polygonal subsets of the plane with edges identified by translations.

## The translation surface of a rational polygon

The following is the translation surface associated to the triangle with angles $\frac{\pi}{8}, \frac{\pi}{4}, \frac{5\pi}{8}$.

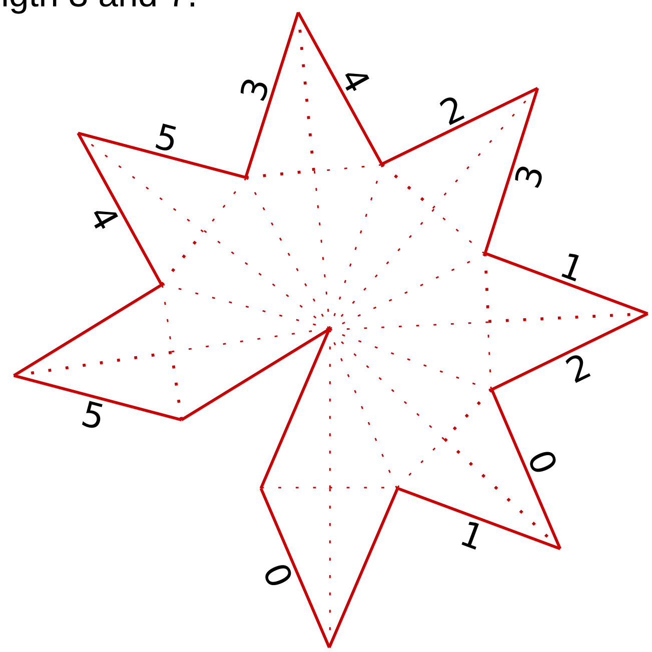



The translation surface associated to a rational polygon is always a closed surface.

There is a natural folding map $f : S(P) \to P$. A periodic billiard path on $P$ (whose orbit type has even period) pulls back to a closed geodesic on $S(P)$.

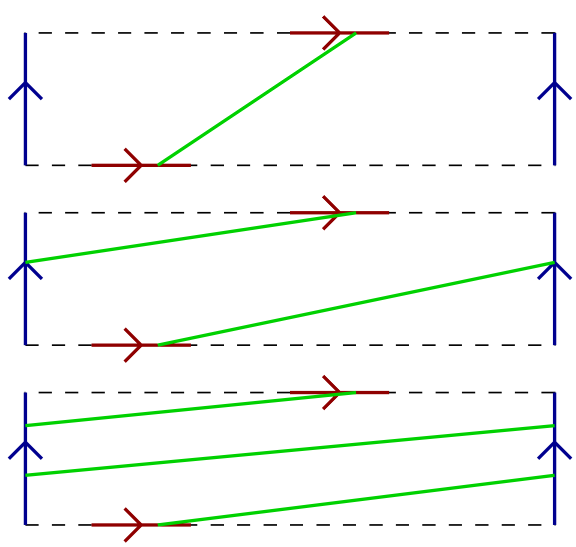
## The translation surface of an irrational polygon

The following is the translation surface associated to the right triangle with legs of length 3 and 7.

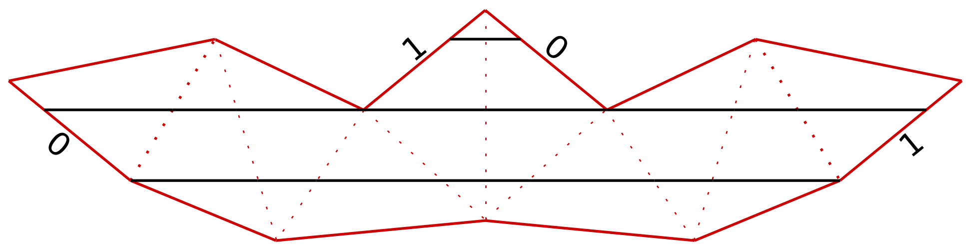



## Getting linear growth

To get linear growth, we look for a cylindrical subsurface of $S(P)$, with some extra identifications.



## We can find such a cylinder inside a translation surface of an irrational polygon.




We conclude that this triangle $T$ satisfies $\liminf_{t\to\infty} \frac{N(T,t)}{t} > 0$.

This configuration is actually stable under small perturbations of the polygon. So, there is an open neighborhood $U$ of $T$ such that

$$\liminf_{t\to\infty} \frac{N(U,t)}{t} > 0.$$

## Affine automorphisms

Let $X$ be a subsurface of a translation surface (with piecewise linear boundary). So $X$ may be described as a union of polygons, $\bigsqcup_i P_i/\sim$, with $\sim$ gluing together some edges by translations.

Let $A \in \mathrm{SL}(2,\mathbb{R})$. $A$ acts linearly on the plane, and takes parallel lines to parallel lines.

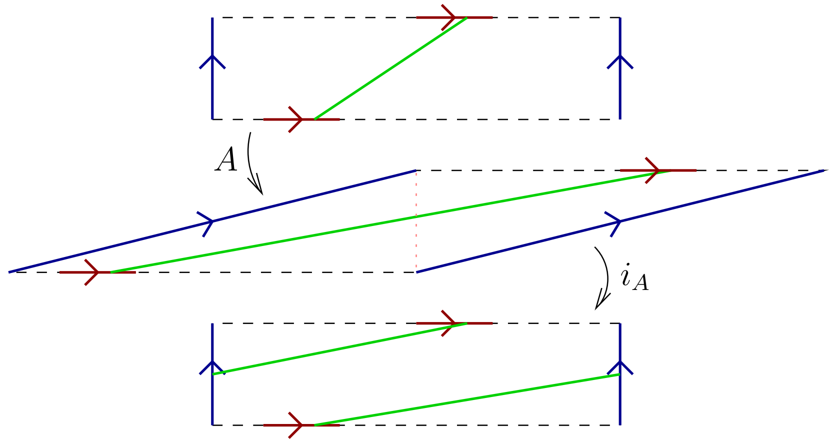
Define **A(X)** $= \bigsqcup_i A(P_i)/\sim$, where $\sim$ identifies the same edges by translation. Also, we use $A$ to denote the map $A : X \to A(X)$.

The **Veech group** of $X$ is the subgroup $\Gamma(X) \subset \mathrm{SL}(2,\mathbb{R})$ consisting of those $A \in \mathrm{SL}(2,\mathbb{R})$ for which there is a direction preserving isometry $i_A : A(X) \to X$.

A composition of the form $i_A \circ A : X \to X$ is known as an **affine automorphism of $X$**.

## Affine automorphisms and our example

Let $c$ be the circumference of our cylinder and $h$ be the height. Let $A = \begin{bmatrix} 1 & c/h \\ 0 & 1 \end{bmatrix}$. Then $A$ is in the Veech group of our subsurface.

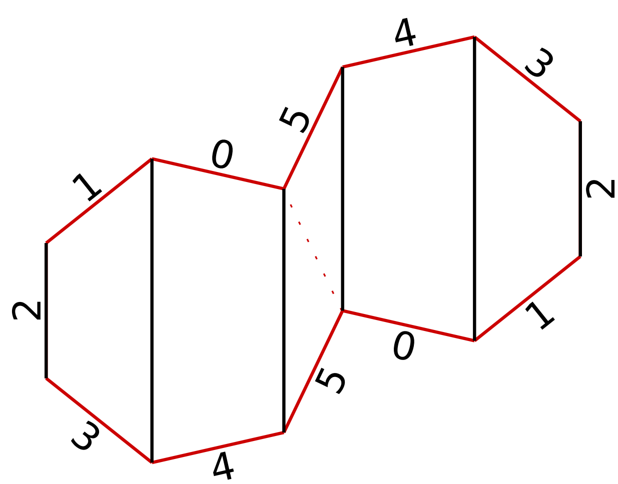



Our list of geodesics is $(i_A \circ A)^n(\gamma)$, where $\gamma$ is our initial geodesic.

## Veech groups of closed surfaces can be large

**Theorem (Veech, 1989)**

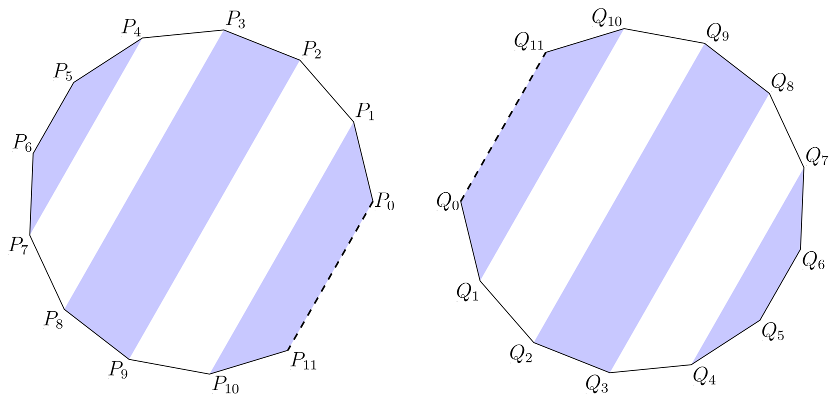
*Fix $n \geq 3$. The Veech groups of the surface built by identifying edges of two regular n-gons is a lattice in $SL(2, \mathbb{R})$.*



## Some special subsurfaces

Let $n \geq 2$ and $\theta < \frac{2\pi}{n}$. Let $z = e^{i\theta}$.

Let $P_i = z^i$ and $Q_i = -z^i$ for $i = 0, \ldots, n$. Let $S(n, \theta)$ denote the translation surface obtained by identifying the convex hull of $\{P_i\}$ to the convex hull of $\{Q_i\}$ and gluing edges $\overline{P_i P_{i+1}}$ to $\overline{Q_{i+1} Q_i}$ by translation.

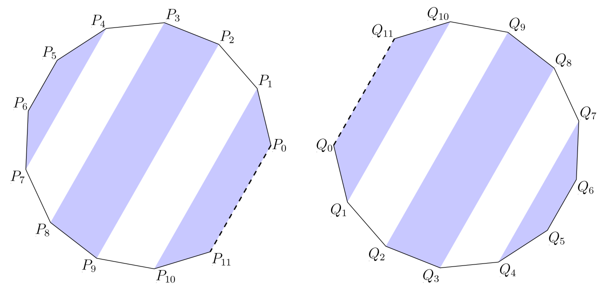



**Proposition**

*The surfaces $S(n, \theta)$ all have a parabolic in their Veech groups.*

## Seeing the parabolic (1)

Our parabolic acts as a single Dehn twist in each of the cylinders depicted.



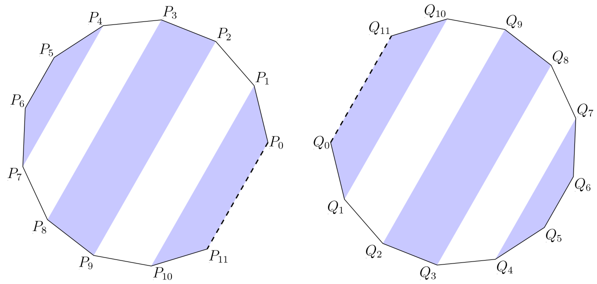

The surface $S(n, \theta)$ consists of a union of cylinders in a fixed direction.

To twist in a horizontal cylinder, we applied the matrix $A = \begin{bmatrix} 1 & c/h \\ 0 & 1 \end{bmatrix}$, where $c$ was the circumference and $h$ was the height of the cylinder.

So, we only need to check that $c/h$ is the same for all cylinders. (This was an observation of Veech.)

## Seeing the parabolic (2)

Recall $z = e^{i\theta}$ and $P_k = z^k$ and $Q_k = -z^k$.




The holonomy vector of the circumference of the $k$-th cylinder is

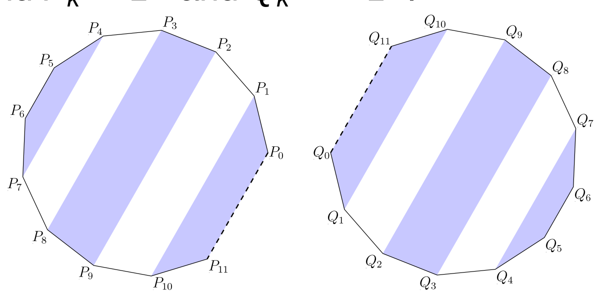
$$\begin{aligned} c_k &= (P_{n-k} - P_k) + (Q_{k+1} - Q_{n-k-1}) \\ &= z^{n-k} - z^k - z^{k+1} + z^{n-k-1} \\ &= (z+1)(z^{n-k-1} - z^k) \end{aligned}$$

An orthogonal vector whose length is the height is

$$\begin{aligned} h_k &= \tfrac{1}{2}((P_{k+1} - P_k) + (P_{n-k-1} - P_{n-k})) \\ &= \tfrac{1}{2}(z^{k+1} - z^k + z^{n-k-1} - z^{n-k}) \\ &= \tfrac{1}{2}(z-1)(z^k - z^{n-k-1}) \end{aligned}$$

## Seeing the parabolic (3)

Recall $z = e^{i\theta}$ and $P_k = z^k$ and $Q_k = -z^k$.




We found $c_k = (z+1)(z^{n-k-1} - z^k)$ and $h_k = \frac{1}{2}(1-z)(z^{n-k-1} - z^k)$.

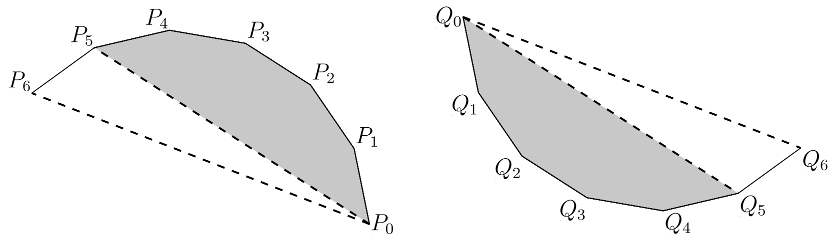
So the length of the circumference of the $k$-the cylinder divided by the height is

$$\left|\frac{c_k}{h_k}\right| = \left|\frac{2(z+1)}{1-z}\right|.$$

This is independent of $k$ so there is a parabolic in the Veech group, whose corresponding affine automorphism simultaneously Dehn twists in each cylinder.

## Building more closed geodesics

The final ingredient is that $S(n-1,\theta)$ embeds into $S(n,\theta)$ as shown below. Let $\iota_n$ denote this inclusion.




Also, let $\tau_n$ denote our affine automorphism $S(n,\theta) \to S(n,\theta)$.

We inductively build a family of geodesics $C_n$ in $S(n,\theta)$ for $n \geq 2$.

- $S(2,\theta)$ is just a cylinder, so define $C_2$ to be the set containing just the core curve.
- $C_n = \{\tau_n^{k_n} \circ \iota_n(\gamma) \mid \gamma \in C_{n-1} \text{ and } k_n \geq 0\}$.

We restrict to $k_n \geq 0$ to guarantee curves are uniquely determined by the sequence of powers $(k_3, \ldots, k_n)$.

## Growth rates

### Proposition

*Fix $\theta < \frac{2\pi}{n}$. Let $N(C_n, t)$ denote the number of paths in the collection $C_n$ of curves in $S(n,\theta)$ whose length is less than $t$. Then for $n \geq 3$,*

$$\liminf_{t\to\infty} \frac{N(C_n,t)}{t \log^{n-3} t} > 0.$$

The proof uses induction and grueling calculus estimates.

### Corollary

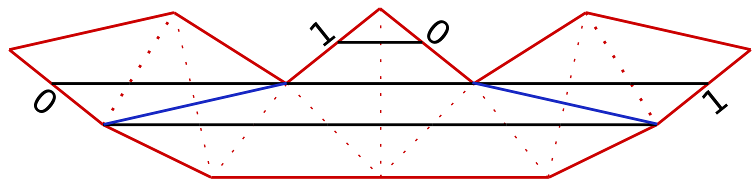
*Let $N\big(S(n,\theta),t\big)$ be the number of closed geodesics in $S(n,\theta)$ of length $< t$ as before. Then* $\liminf_{t\to\infty} \frac{N(S(n,\theta),t)}{t \log^{n-3} t} > 0.$

### Question

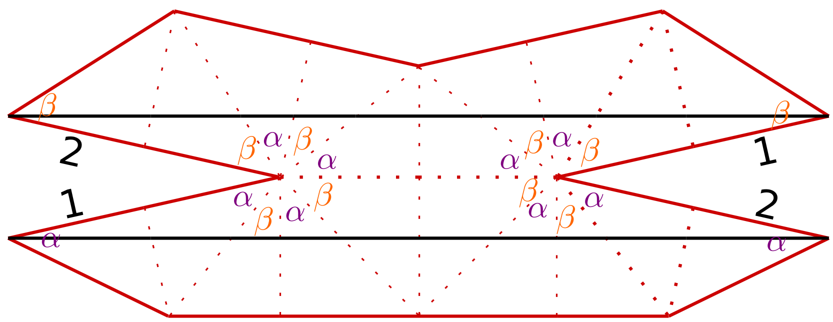
*Are there translation surfaces with boundary of genus $g$ where $N\big(S(n,\theta),t\big)$ grows faster than $t \log^{g-1} t$?*

## Finding $S(n,\theta)$ in translation surfaces of irrational polygons

Here was our cylinder from before.

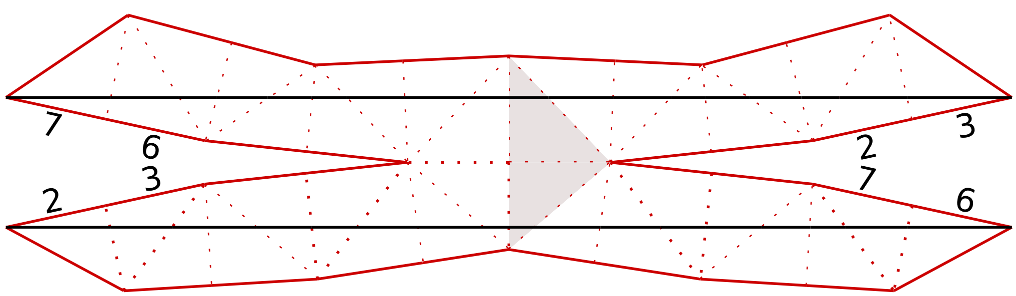



To make the surface easier to see, we cut along the altitudes of our triangle and reglue. This surface is $S(3, \pi - \alpha - \beta)$ in disguise!

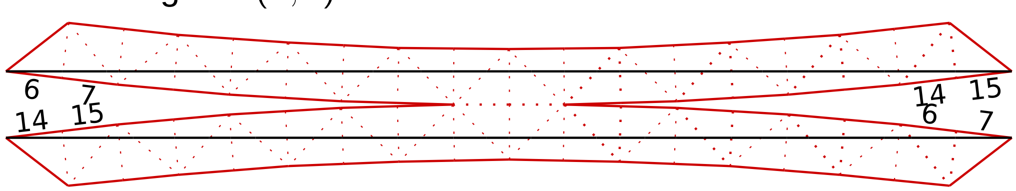



## Finding $S(5,\theta)$

An embedding of $S(5,\theta)$.




An embedding of $S(9,\theta)$.




As a sequence of acute triangles approaches the 45-45-90 triangle, you eventually see every $S(n,\theta)$.

## Concluding remarks

A **stretched limit** can be taken of the surfaces $S(n, \theta)$ as $n \to \infty$ and $\theta \to 0$ appropriately.

The result is an infinite area surface built from the convex hull of $\{(n, n^2) \mid n \in \mathbb{Z}\}$ and $\{(n, -n^2) \mid n \in \mathbb{Z}\}$.