Billiards in nearly isosceles triangles

(joint work with Rich Schwartz)

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DePaul Analysis Seminar

October 1st, 2008

Preprint available at:

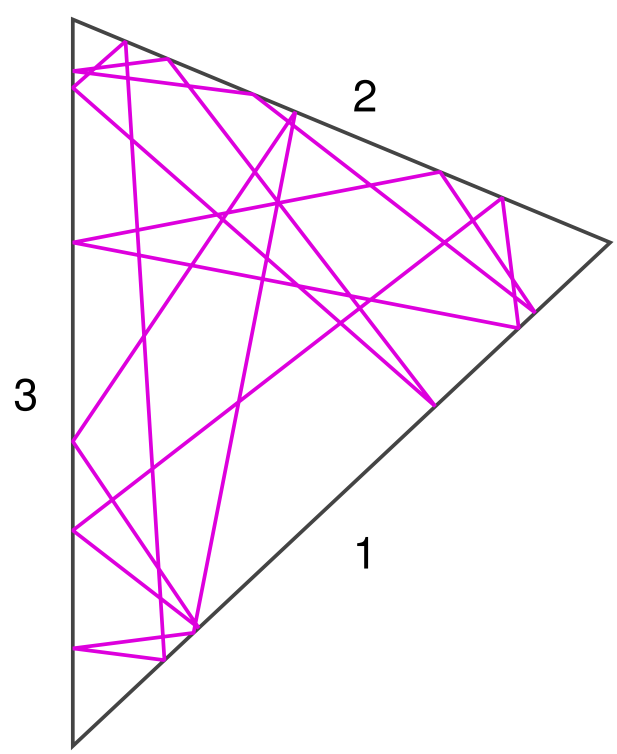
- `http://www.math.northwestern.edu/~wphooper/docs/`
- `http://arxiv.org/abs/0807.3498`

Billiard trajectories in polygons

A **billiard path** in a polygon P is a unit speed path which is

- geodesic in the interior,
- and bounces off ∂P according to the laws of optics. (The angle of incidence must equal the angle of reflection.)

Billiard trajectories are not defined through vertices.



Motivating open question: Periodic billiard paths

A **periodic billiard path** is a billiard path which returns to its starting point traveling in the same direction.

Open Question:
Does every polygon have a periodic billiard path?

The following classes of polygons are known to have periodic billiard paths.

- Acute triangles (Fagnano, 1755)
- Rational polygons (polygons whose angles are rational multiples of π) (Masur, 1986)
- Triangles whose largest angle is less than 100 degrees (Schwartz, 2005)

Main result

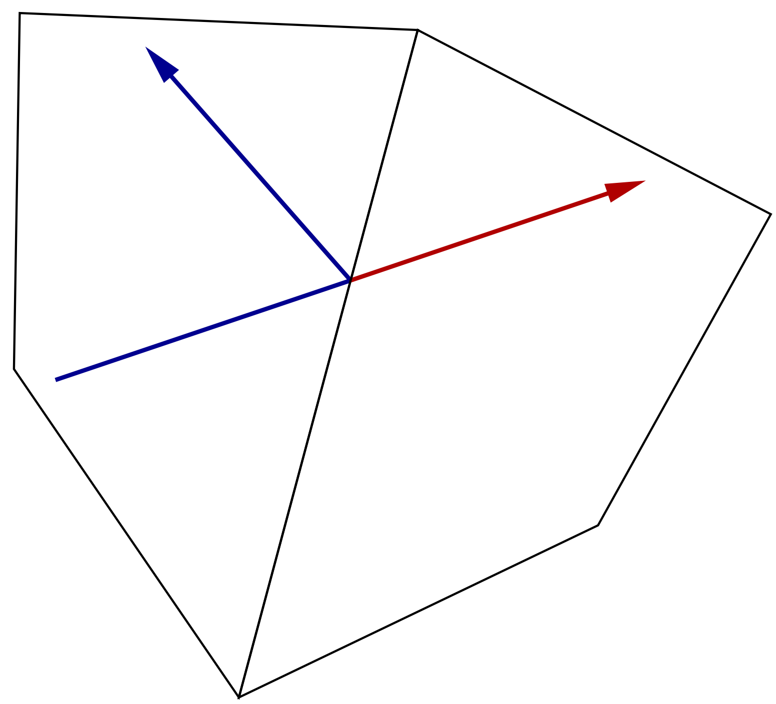
Today's main result is that nearly isosceles triangles have periodic billiard paths.

More rigorously,

Theorem (H.-Schwartz):
There is an open set U of triangles containing the isoceles triangles such that every triangle $T \in U$ has a periodic billiard path.

Proving a polygon has a periodic billiard path

Consider the simplifying observation to the below.



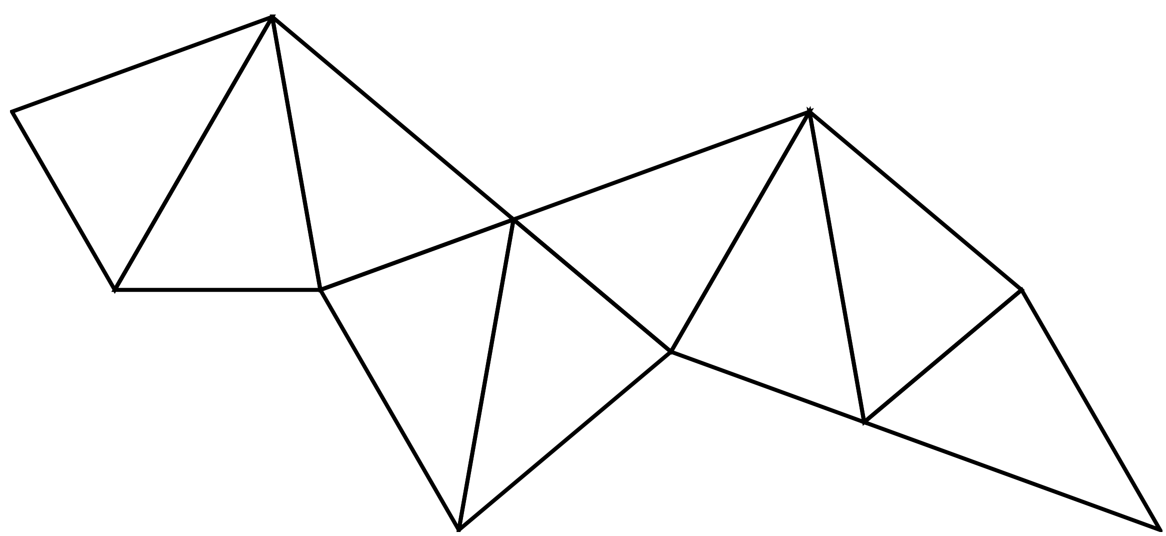
- Suppose a billiard ball is aimed toward the edge of a polygon.
- Reflect the polygon across the edge.
- If we allow the billiard ball to pass straight through the edge, then its location in the new polygon is the same as its location in the original polygon if we had followed the billiard trajectory.

Unfolding a polygon

- An *orbit-type* W is a sequence of edges of a polygon.

- The sequence of edges hit by a periodic billiard path is called the *orbit-type* of the path.

- The unfolding of a polygon P, $U(P, W)$, is the chain of polygons obtained by iteratively reflecting the polygon P across the edges in the sequence W.

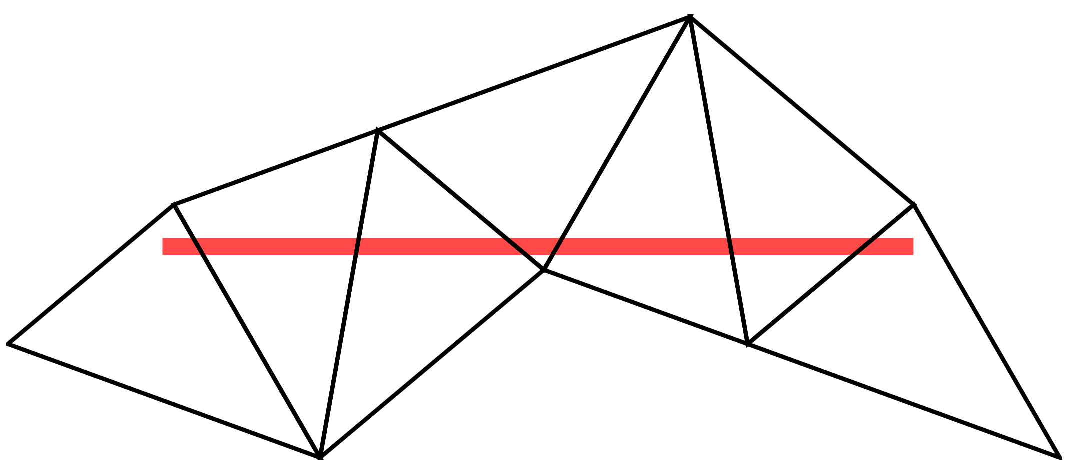


Conditions for the existence of a periodic billiard path

The following two conditions on the unfolding $U(P, W)$ are necessary and sufficient to guarantee the existence of billiard path which hits the edges of P in the sequence W of even length.

1. The last triangle of the unfolding must be a parallel translate of the first triangle of the unfolding.

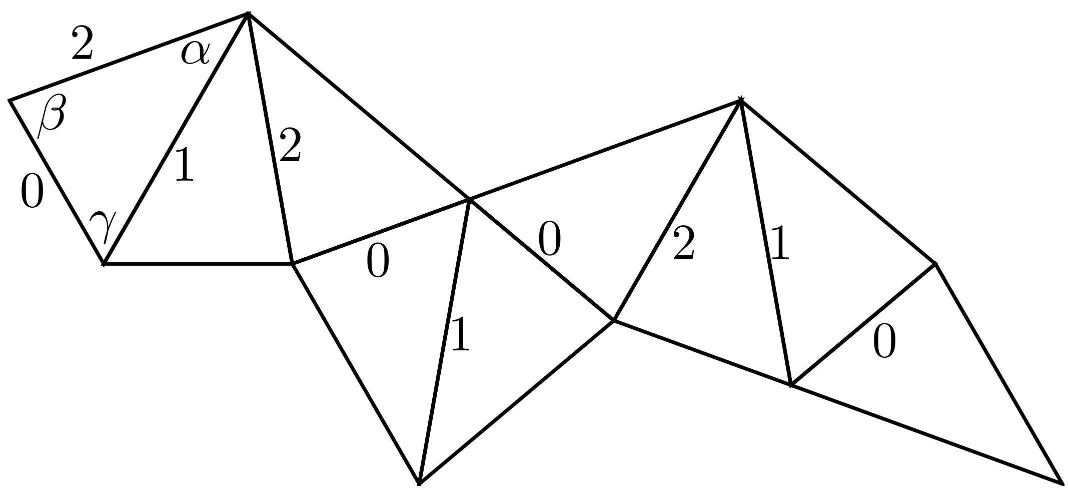
2. There must be line segment contained entirely in the unfolding (and not passing through vertices) which runs from a point in the initial triangle to the corresponding point in the final triangle.



The closed condition for existence

Let Δ be a triangle with angles α, β, and γ.

Claim: Let W be a sequence of edges of Δ of even length. Then the last polygon of $U(\Delta, W)$ is a rotation of the first by a linear combination of $\{2\alpha, 2\beta, 2\gamma\}$.



The unfolding of the orbit-type $W = 12010210$.

Proof. $01 \mapsto +2\gamma \qquad 12 \mapsto +2\alpha \qquad 20 \mapsto +2\beta$

$10 \mapsto -2\gamma \qquad 21 \mapsto -2\alpha \qquad 02 \mapsto -2\beta$ □

Proposition. This linear sum is a multiple of 2π on an open set if and only if for each $d \in \{0, 1, 2\}$, the number of times d appears in odd positions of W equals the number of times in even positions.

The tile

Definition: The **orbit-tile** $O(W)$ of an orbit-type W is the set of all triangles Δ for which there is a periodic billiard path in Δ that hits the edges of a triangle according to the sequence W.

We have seen that $O(W)$ is a non-empty open set only if the number of times $d \in \{0, 1, 2\}$ appears in odd positions of W equals the number of times it appears in even positions. In this case, we call W **stable**.

Otherwise $O(W)$ is an open subset of a union of "rational lines", those places where

$$2a\alpha + 2b\beta + 2c\gamma \equiv 0 \pmod{2\pi},$$

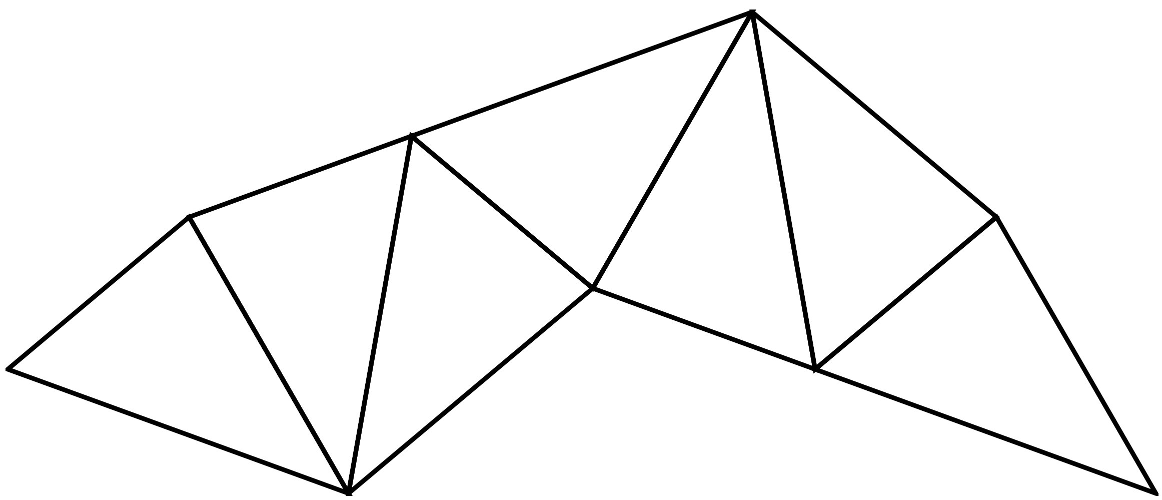
where $a, b, c \in \mathbb{Z}$ are not all equal. In this case, W is called **unstable**.

To prove large sets of triangles have periodic billiard paths, we will need to consider stable W.

The Fagnano curve

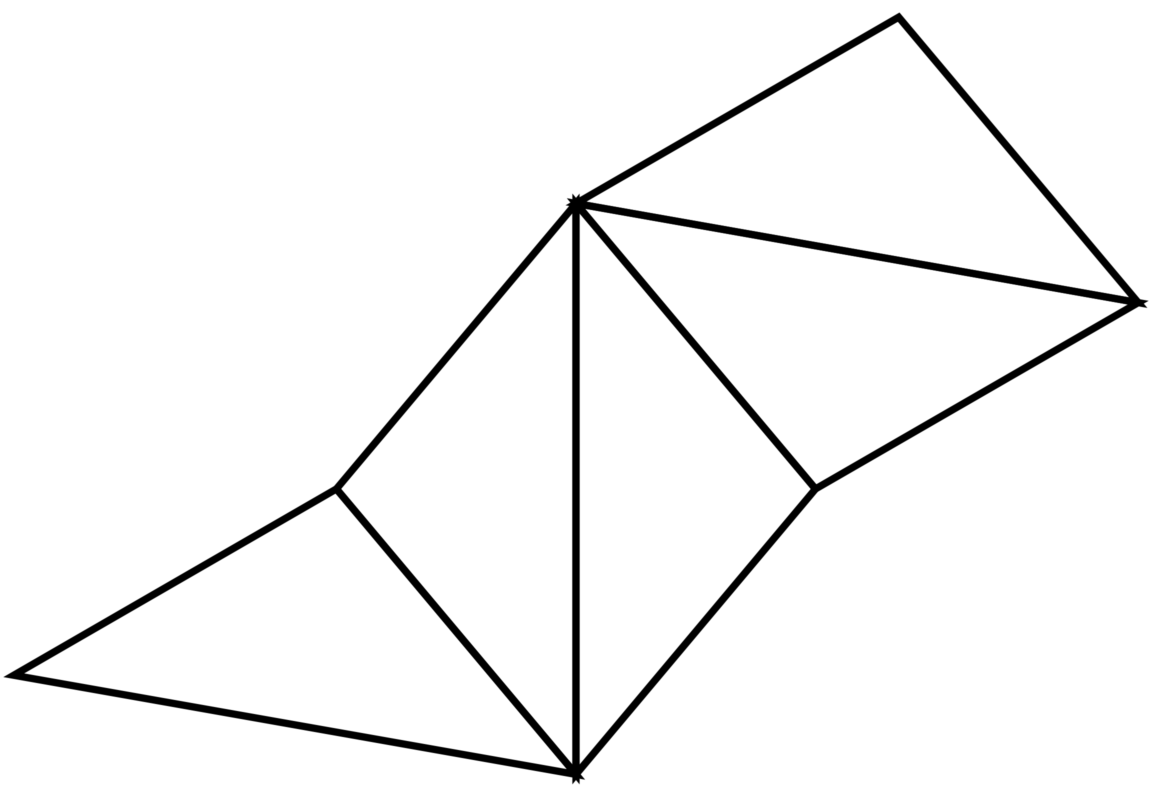
The Fagnano curve is the simplest example of a stable periodic billiard path.

Claim. Let $W = 012012$. The corresponding tile $O(W)$ is the collection of all acute triangles.



An unstable periodic billiard path

All isosceles triangles have an unstable periodic billiard path corresponding to the orbit-type $W = 1020$. The corresponding tile is precisely the set of isosceles triangles.



The 45-45-90 triangle (1)

The 45-45-90 triangle is slightly difficult, because of the following theorem.

Theorem (H). Right triangles do not have stable periodic billiard paths.

However, Schwartz's theorem implies that a neighborhood of the 45-45-90 triangle has periodic billiard paths.

Theorem (Schwartz). Every triangle whose largest angle is less than 100 degrees has a periodic billiard path.

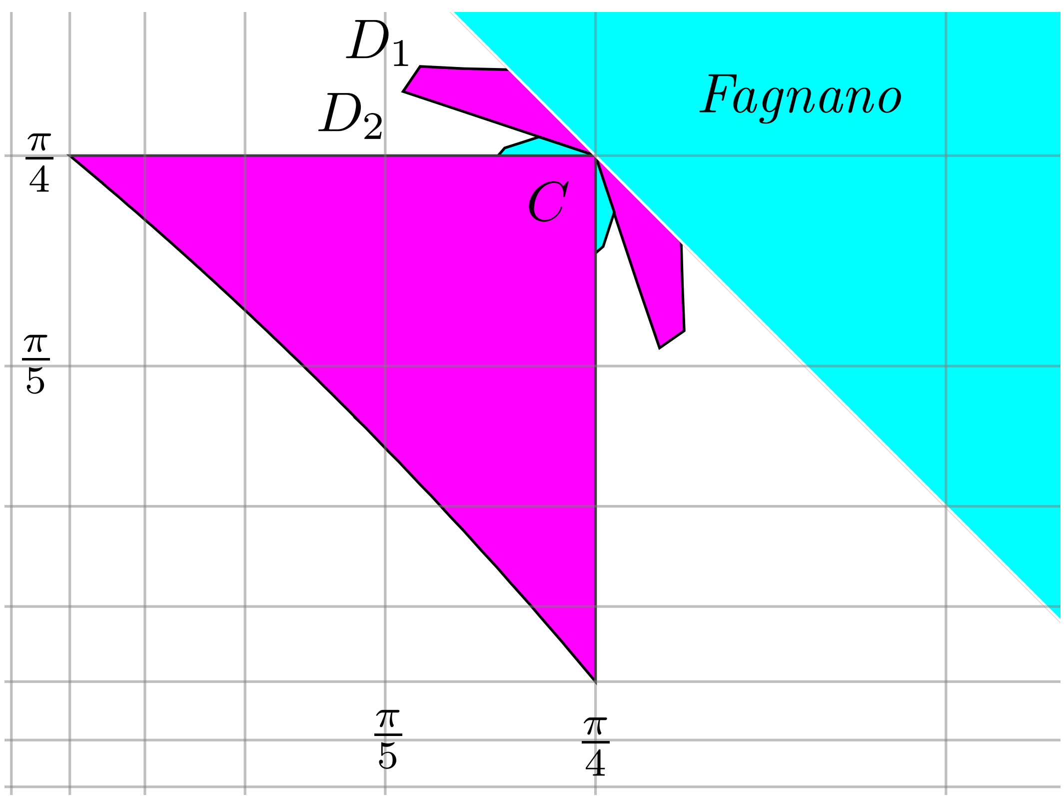
The 45-45-90 triangle (2)

Five tiles cover all but 6 rays leaving the 45-45-90 triangle. Define the orbit types

$$C = (1202010)^2$$

$$D_1 = 201020120201020120202102010202102010$$

$$D_2 = \begin{matrix} 20102010201202010201020120 \\ 20210201020102021020102010 \end{matrix}$$



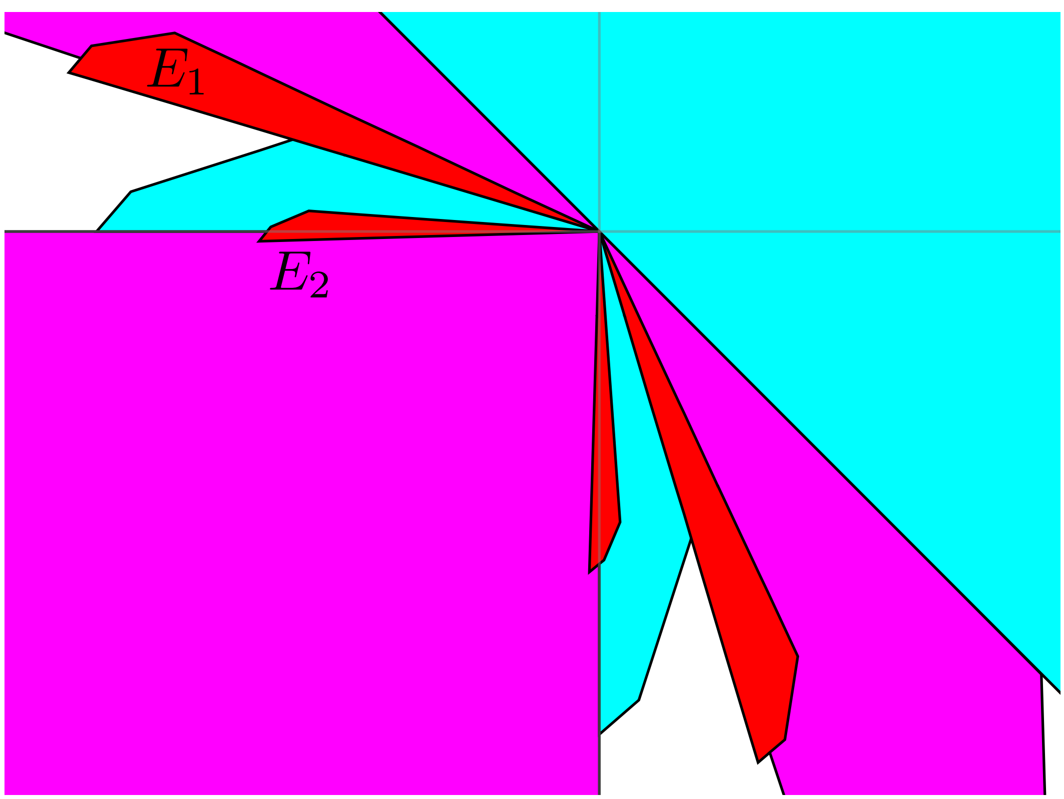
The 45-45-90 triangle (3)

Right triangles all have periodic billiard paths.

The remaining 4 rays are covered by four tiles. Define

$$E_1 = \begin{array}{l} 1202010201202021020102 \\ 0102021020102010201020 \end{array}$$

$$E_2 = \begin{array}{l} 120201020102012020210201020102 \\ 010202102010201020102010201020 \end{array}$$



Almost all isosceles triangles

Let $\triangle_x$ be the isosceles triangle with two angles of measure x.

We have already proved that the set of all acute and right isoscles triangles, $\{\triangle_x \ : \ 0 < x \leq \frac{\pi}{4}\}$, is contained in an open set of triangles which contain periodic billiard paths. (In fact, we only needed finitely many orbit-types.)

Consider the words

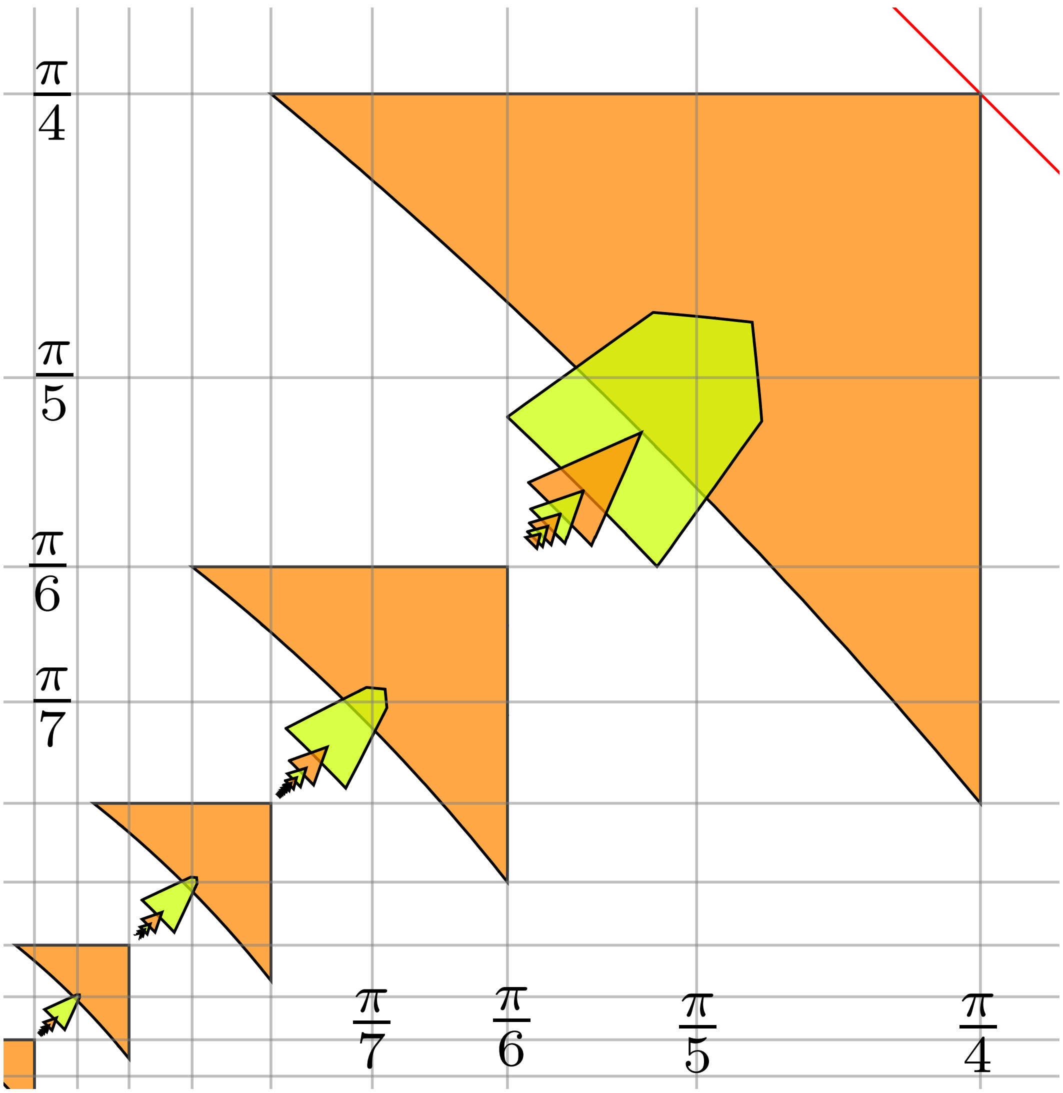
$W_n = (20)^{n-1}(21)^{n-1} \qquad Y_{n,m} = \Big(0(W_n)^m 21\Big)^2.$

Note that $Y_{n,m}$ is stable, while W_n is unstable.

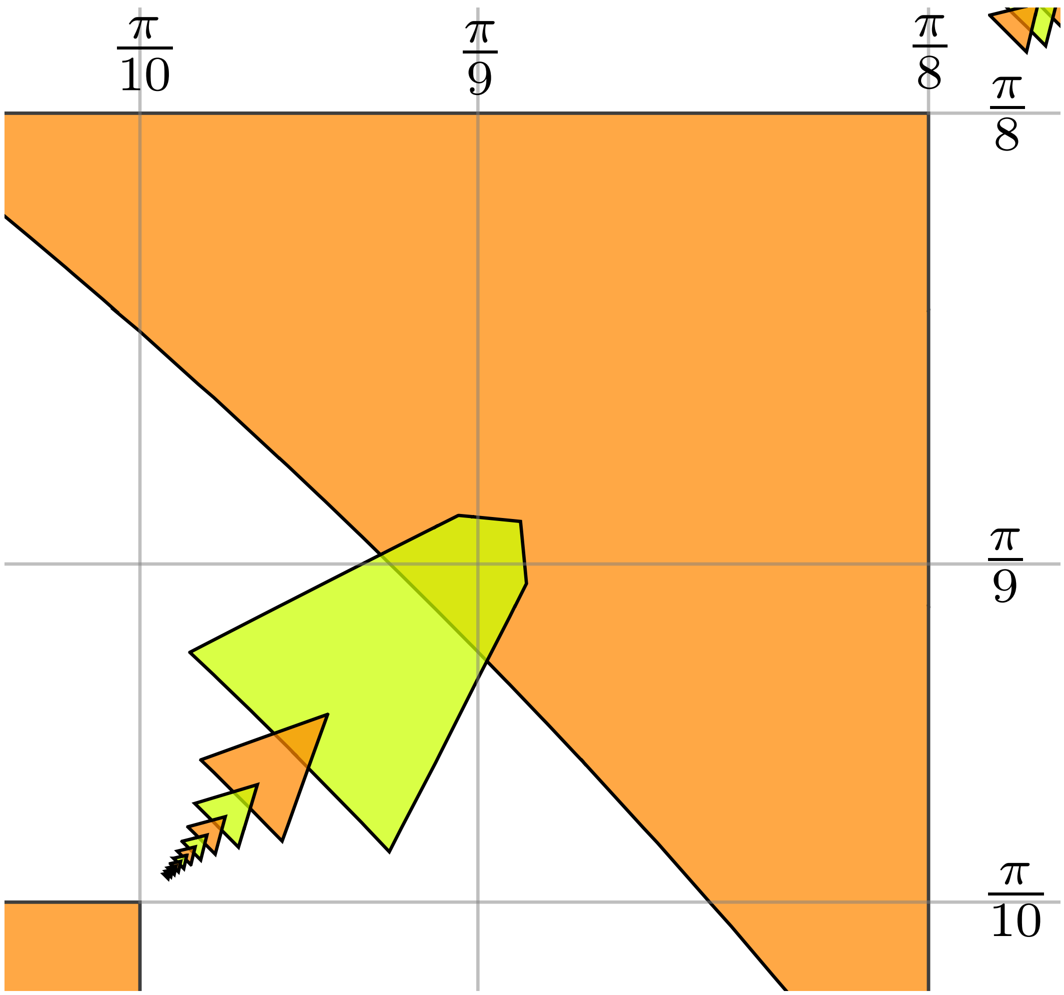
Theorem. *For every integer $n \geq 2$,*

$$\{\triangle_x \ : \ \frac{\pi}{2n+2} < x < \frac{\pi}{2n}\} \subset \bigcup_{m=1}^{\infty} O(Y_{n,m})$$

In other words, all obtuse isosceles triangles other than $V_k = \triangle_{\frac{\pi}{2k}}$ for $k > 2$ lie in an open orbit-tile.

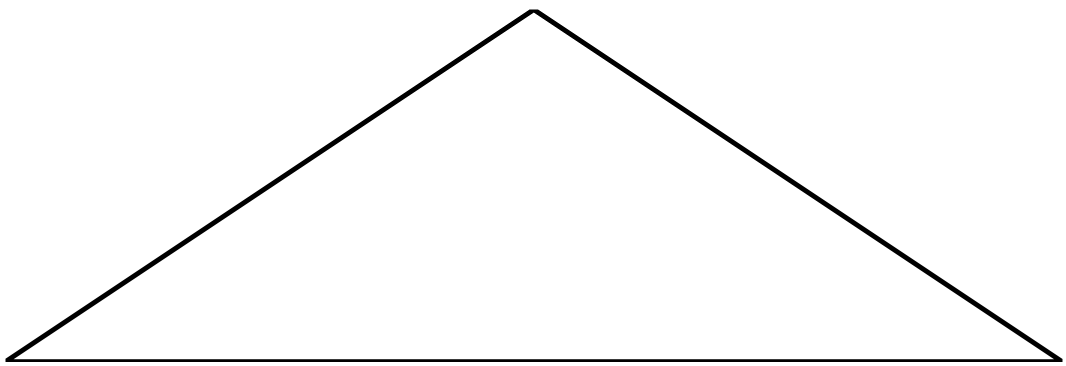

$\frac{\pi}{4}$
$\frac{\pi}{5}$
$\frac{\pi}{6}$
$\frac{\pi}{7}$
$\frac{\pi}{7}$
$\frac{\pi}{6}$
$\frac{\pi}{5}$
$\frac{\pi}{4}$

Close up of the tiles $O(Y_{4,m})$ for $m = 1, 2, 3, \ldots$

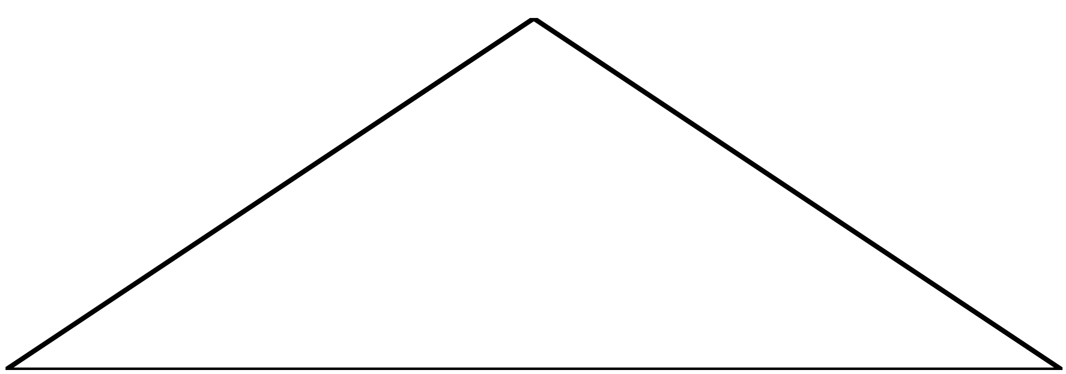


Symmetry and isosceles triangles

Proposition. Any billiard path which hits the midpoint of the long side of an obtuse isosceles triangle twice closes up.



Proposition. Any billiard path which starts out parallel to the long side of an obtuse isosceles triangle closes up and hits the midpoint of the long side closes up.

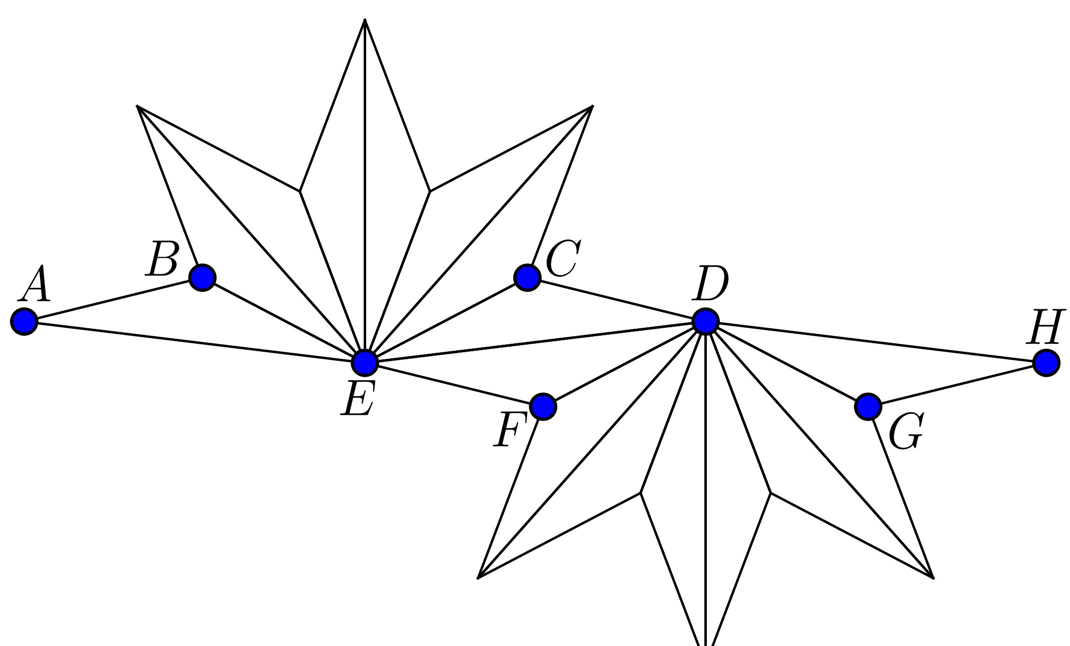


The unstable orbit-types W_n

Recall our unstable words $W_n = (20)^{n-1}(21)^{n-1}$.

Proposition. The orbit type W_n describes a periodic billiard path in each isosceles triangle Δ_x with $x < \frac{\pi}{2n-2}$.

Proof. The following is the unfolding for W_5.

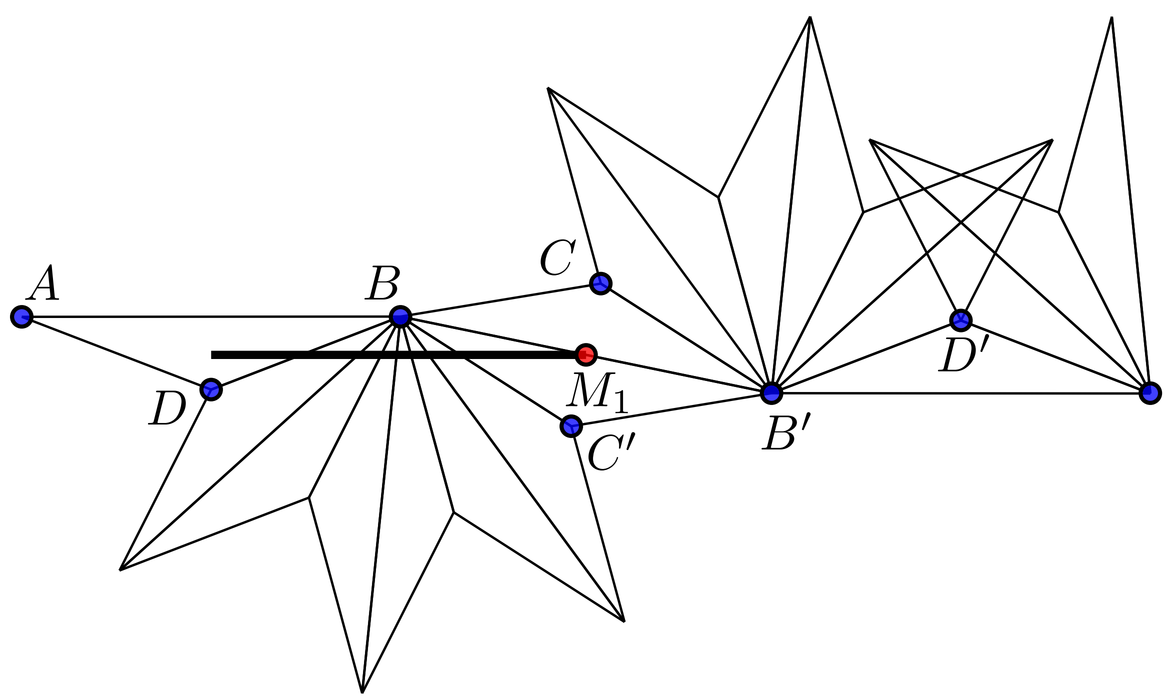


The stable orbit-type $Y_{n,1}$

Recall, $Y_{n,1} = \left(0W_n21\right)^2 = \left(0(20)^{n-1}(21)^{n-1}21\right)^2$.

Lemma. The stable orbit type $Y_{n,1}$ describes a periodic billiard path in each isosceles triangle Δ_x with $\frac{\pi}{2n+1} \leq x \leq \frac{\pi}{2n}$.

Proof. The following is the unfolding for a little more than the first half of $Y_{4,1}$.

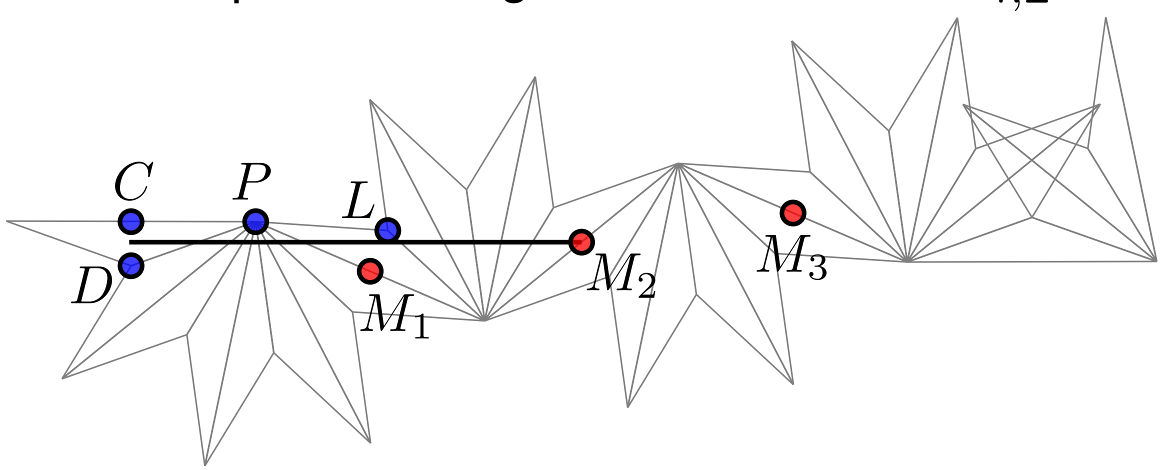


The stable orbit-type $Y_{n,m}$ (1)

Recall, $Y_{n,m} = \left(0(W_n)^m 21\right)^2$, where $W_n = (20)^{n-1}(21)^{n-1}$.

Lemma. For each x with $\frac{\pi}{2n+2} < x < \frac{\pi}{2n+1}$, there is a periodic billiard path in Δ_x with stable orbit-type $Y_{n,m}$ for some $m > 1$.

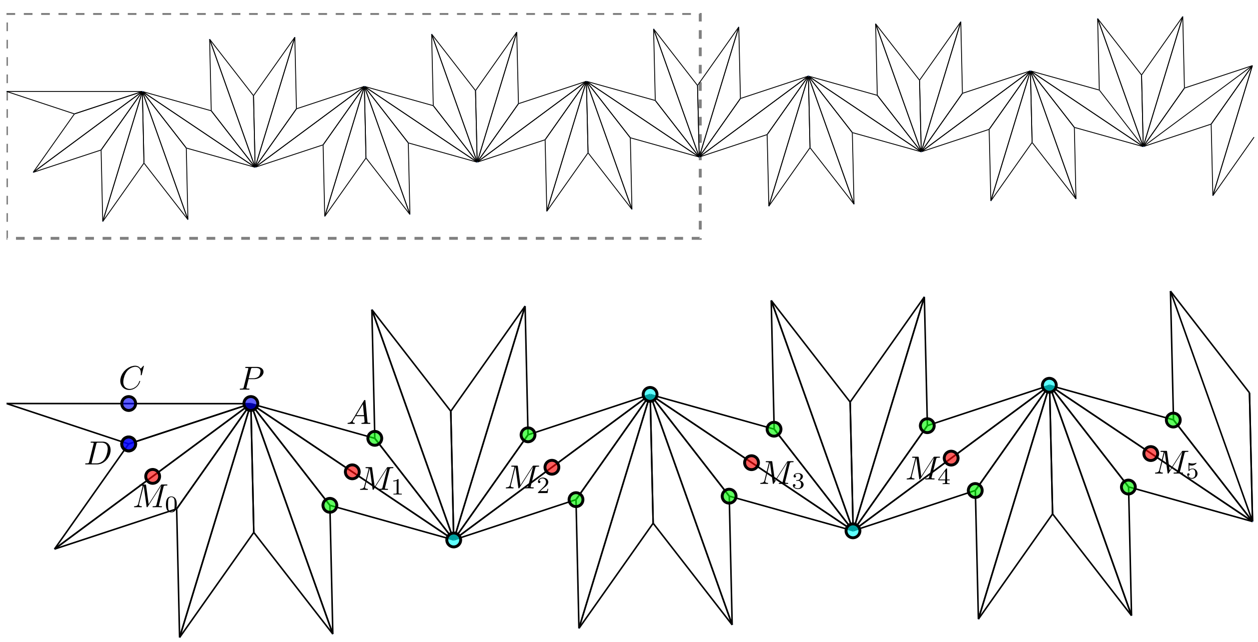
Example unfolding of the first half of $Y_{4,2}$.



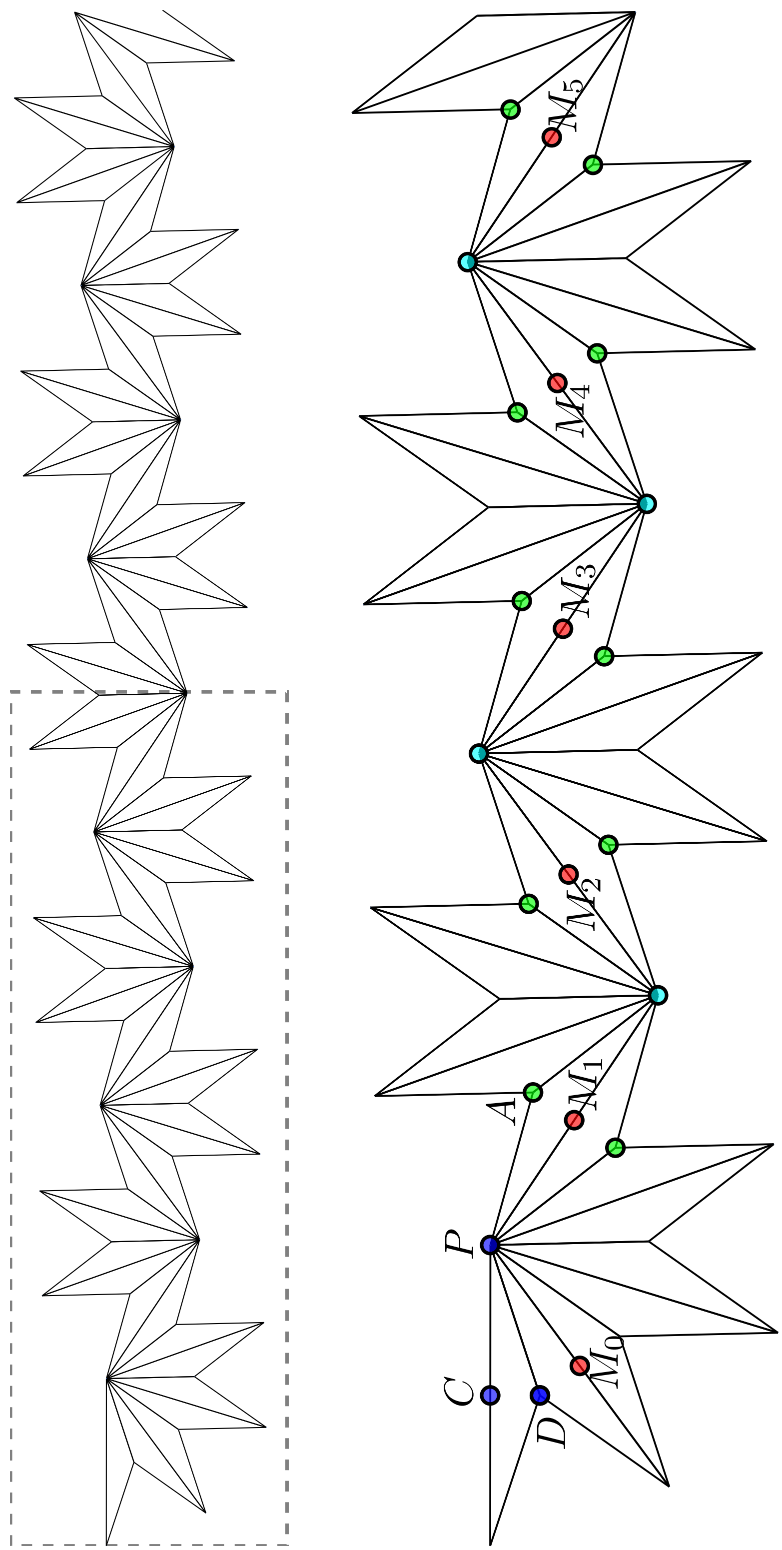
Hitting midpoint M_2 results in a periodic billiard path with orbit type $Y_{n,2}$.

The stable orbit-type $Y_{n,m}$ (2)

Let x satisfy $\frac{\pi}{2n+2} < x < \frac{\pi}{2n+1}$. Consider the unfolding $U(\Delta_x, Y_{n,\infty})$ of the infinite word $Y_{n,\infty} = 1(W_n)^\infty$.

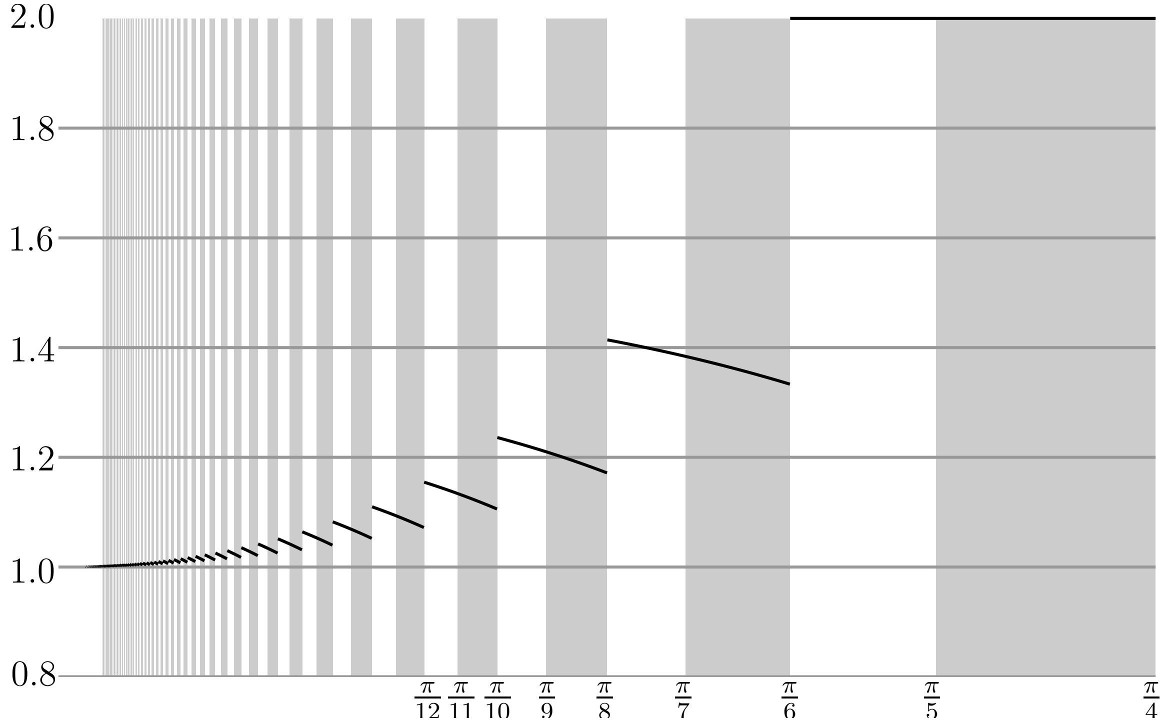


If we can find a horizontal beam through the unfolding from the first triangle which hits the midpoint M_m, then $\Delta_x \in O(Y_{n,m})$.


M_5
M_4
M_3
M_2
M_1
A
P
M_0
C
D

Ratio graph

The following graph shows the ratio of the width of the "beam" to the vertical displacement of the midpoints M_i, for values of x. The relevant values are when $\frac{\pi}{2n+2} < x < \frac{\pi}{2n+1}$ (the white regions).



This suggests that the tiles do not overlap much at all as $x \to 0$. Therefore, we expect that our open neighborhood of the isosceles line gets quite thin as $x \to 0$.

Bad behavior at V_{2^k}

The only triangles that remain are the isosceles triangles V_n, the triangle with two angles of measure $\frac{\pi}{2n}$.

Remark The triangles V_n have Veech's lattice property. Billiards in these polygons are very well understood.

Theorem (H.-Schwartz).

- For $k = 3, 4, 5, \ldots$, the triangle V_{2^k} does not lie in the interior of an orbit tile.

- For $n \geq 3$ and not a power of two, V_n does lie in in the interior of an orbit tile.

With this theorem, the only triangles that remain are the triangles V_{2^k} for $k \geq 3$.

Worse behavior at V_{2^k}?

Theorem (Schwartz). No open neighborhood 30-60-90 triangle has a finite covering by orbit-tiles.

Conjecture. For each $k \geq 3$, no open neighborhood of the triangle V_{2^k} has a finite covering by orbit-tiles.

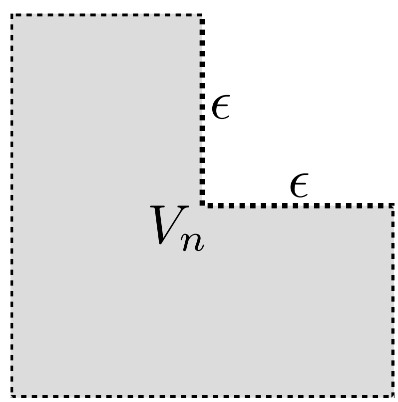
Rephrasing:
Normalize each triangle to have area one. Consider the function L that assigns the length of the shortest periodic billiard path to a given triangle.

The above are equivalent to the statement “L is not locally finite at the given point.”

Covering part of the neighborhood of V_n

Let $\Delta(\alpha, \beta)$ denote the triangle with acute angles α and β. Let

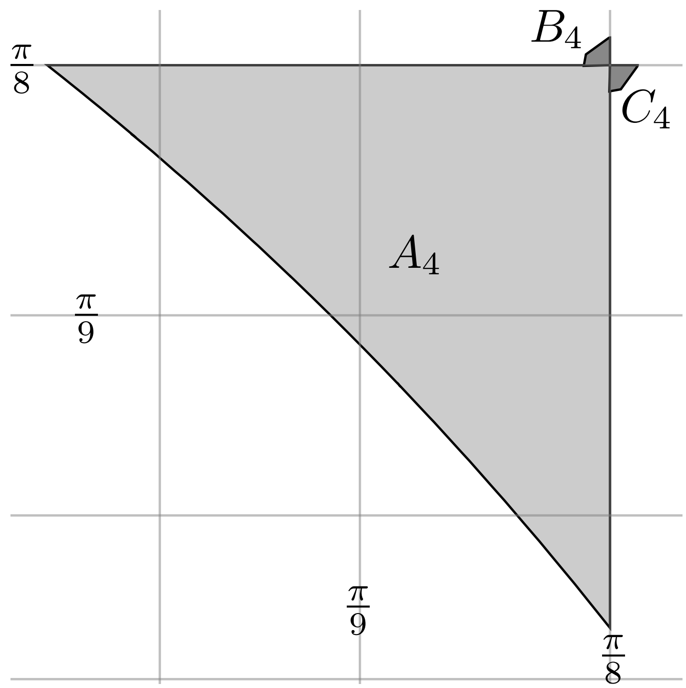
$$U_n(\epsilon) = \left\{ \Delta(\frac{\pi}{n} + a, \frac{\pi}{n} + b) \;:\; \begin{array}{l} |a| < \epsilon, |b| < \epsilon, \\ \text{and } a < 0 \text{ or } b < 0 \end{array} \right\}.$$

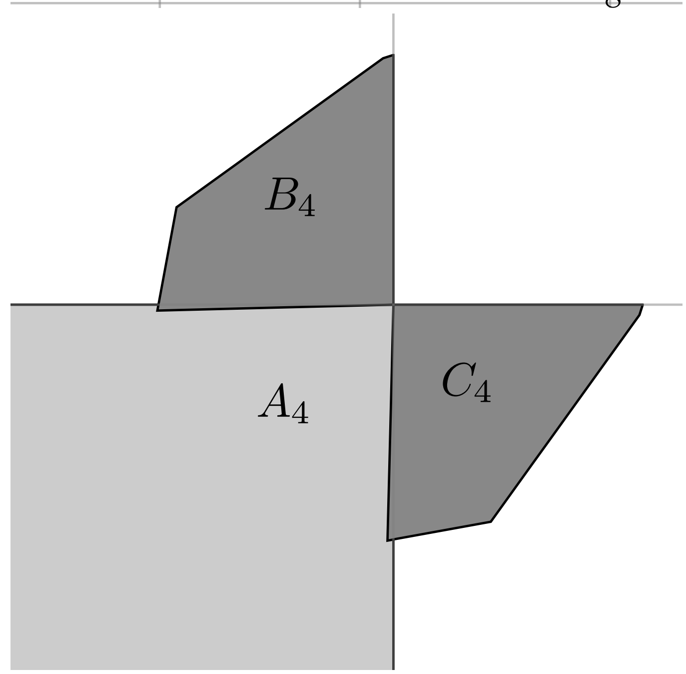


Let $A_n = \Big((20)^n (10)^{n-1} 1\Big)^2$. There are longer sequences of words B_n and C_n, which are too long to list here, such that the following is satisfied. (Eg. $|B_n| = 40n - 60$.)

Proposition. For all n there is an $\epsilon > 0$ such that $U_n(\epsilon) \subset O(A_n) \cup O(B_n) \cup O(C_n)$.

The following illustrate the covering of an $U_4(\epsilon)$ by with tiles $O(A_4)$, $O(B_4)$ and $O(C_4)$.

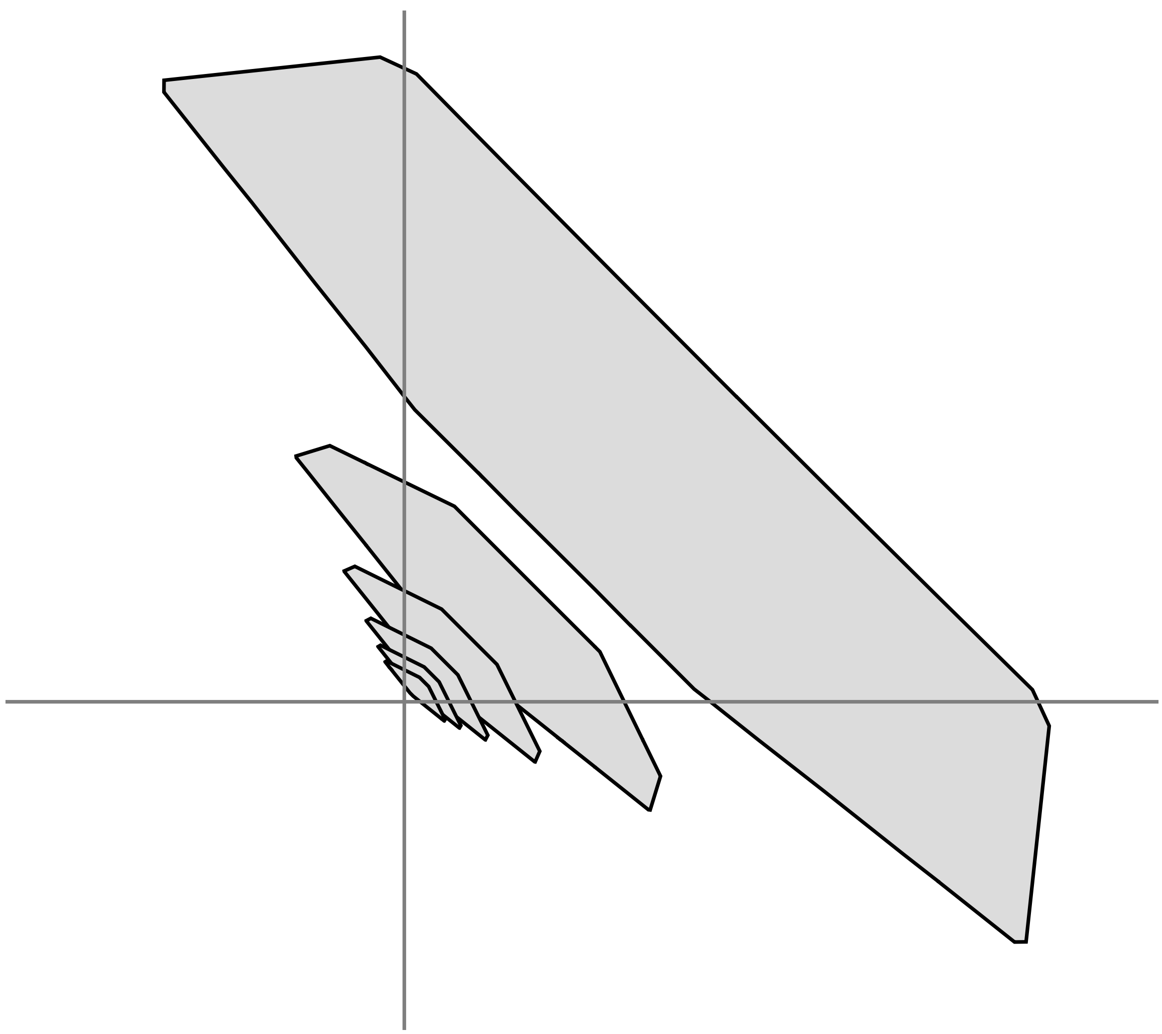




The remaining quadrant (1)

Let $N_n(\epsilon) = \{\Delta(\frac{\pi}{n} + a, \frac{\pi}{n} + b) \ : \ 0 \leq a, b < \epsilon\}$.

Theorem. There is a sequence of stable words $W_{n,k}$ (with $|W_{n,k}| = 24n + 30k^2 - 68k - 20$) such that $\forall n \geq 3 \ \exists \epsilon > 0$ such that $N_n(\epsilon) \subset \bigcup_k O(W_{n,k})$.



The remaining quadrant (2)

Theorem Let $\phi_{n,k}$ be the dilation which maps V_n to 0 and expands distances by

$$\zeta_n k^2 \text{ with } \zeta_n := 2(n-1)\cot(\pi/2n) \approx 4\pi n^2$$

If n is held fixed and $k \to \infty$ then the closure of $\phi_{n,k}\Big(O(W_{n,k})\Big)$ Hausdorff-converges to the convex quadrilateral Q_n with vertices

$$(-\frac{1}{n}, 1-\frac{1}{n}); \quad (1-\frac{1}{n}, -\frac{1}{n}); \quad (a_n, a_n); \quad (\lambda_n a_n, \lambda_n a_n);$$

where

$$a_n = \frac{1}{2} - \frac{1}{2n}; \quad \lambda_n = \frac{1}{2} - \frac{\tan^2(\pi/2n)}{2}.$$

The convergence is such that any compact subset $Q'_n \subset Q_n$ is contained in $\phi_{n,k}\Big(O(W_{n,k})\Big)$ for k sufficiently large in comparison to n.