On the stability of periodic billiard paths in triangles

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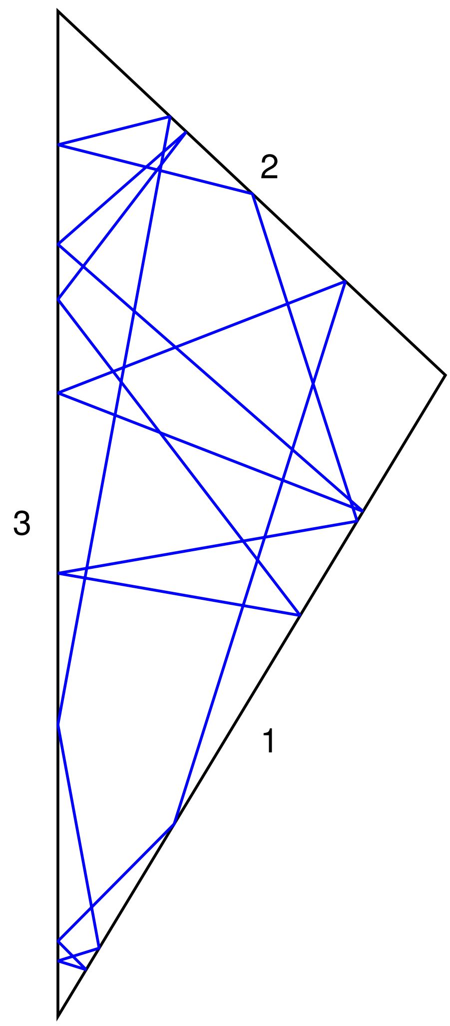
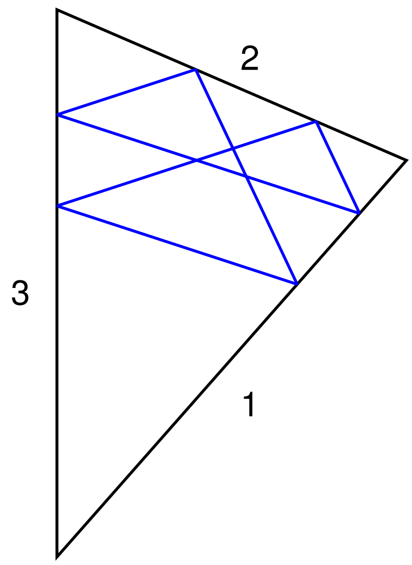


- A periodic billiard path $\widehat{\gamma}$ in a triangle gives rise to the bi-infinite periodic sequence of marked edges it hits.

- We call this sequence the **symbolic dynamics of $\widehat{\gamma}$** and denote it by $s_{\widehat{\gamma}}$.



A periodic billiard path $\widehat{\gamma}$ with symbolic dynamics $s_{\widehat{\gamma}} = \overline{123123}$.

- Let $\mathcal{T}$ be the space of marked triangles up to similarity.

- A periodic billiard path $\widehat{\gamma}$ in a triangle $\Delta \in \mathcal{T}$ is **stable** if there is an open set $U \subset \mathcal{T}$ containing Δ, so that every $\Delta' \in U$ has a periodic billiard path $\widehat{\gamma}'$ with the same symbolic dynamics ($s_{\widehat{\gamma}} = s_{\widehat{\gamma}'}$).

Why do we care about stable periodic billiard paths in triangles?

Open Question. *Does every triangle have a periodic billiard path?*

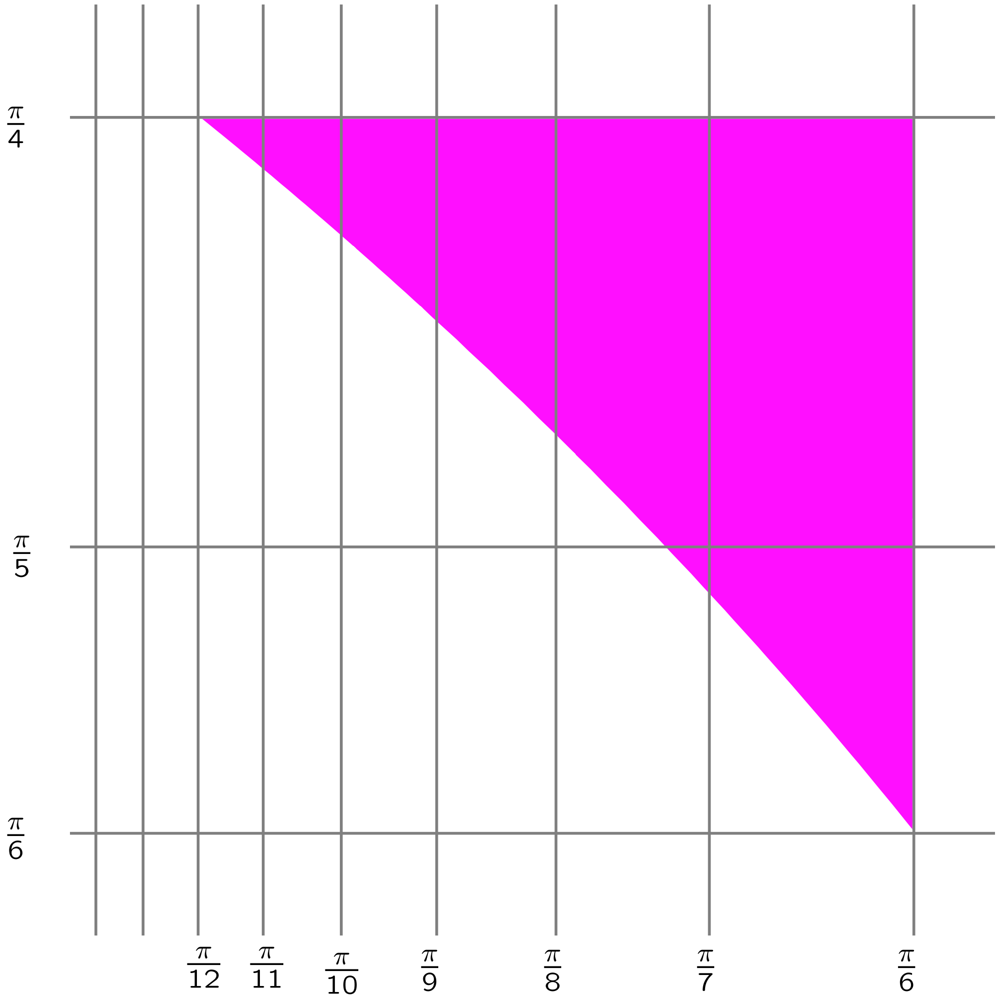
Open sets are useful for covering $\mathcal{T}$. Furthermore,

Proposition. *If the angles (in radians) of the triangle* Δ *are linearly independent over the integers, then all periodic billiard paths in* Δ *are stable.*

For proof see

- Tabachnikov's book *Billiards* (1995),
- Vorobets, Gal′perin, and Stëpin's article *Periodic billiard trajectories in polygons: generating mechanisms* (1991), or
- my thesis

The **tile** of a periodic billiard path $\widehat{\gamma}$ is the subset *tile*$(\widehat{\gamma}) \subset \mathcal{T}$ consisting of all triangles $\Delta \in \mathcal{T}$ with periodic billiard paths $\widehat{\eta}$ with the same symbolic dynamics as $\widehat{\gamma}$ ($s_{\widehat{\gamma}} = s_{\widehat{\eta}}$).



Open Question. *Which triangles admit stable periodic billiard paths?*

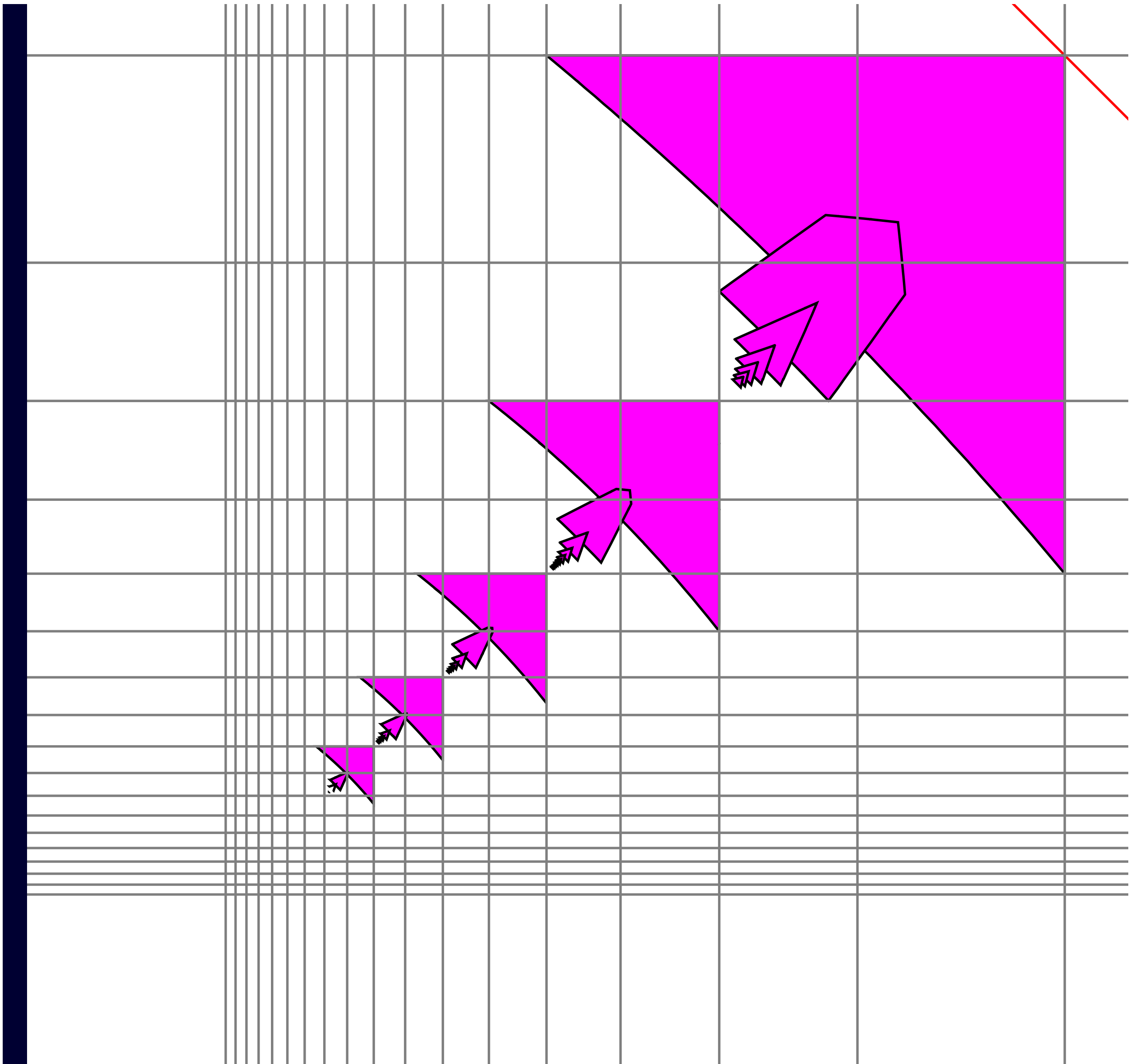
Theorem (Fagnano). *Each acute triangle has a stable periodic billiard path* $\widehat{\gamma}$ *with* $s_{\widehat{\gamma}} = \overline{123}$.

Theorem (Schwartz). *Obtuse triangles with largest angle less than 100 degrees have stable periodic billiard paths.*

Theorem (H (thesis)).

- *All but countably many isosceles triangles have stable periodic billiard paths.* $(\frac{\pi}{2n}, \frac{\pi}{2n}, \frac{(n-1)\pi}{n})$

- *There exist countably many isosceles triangles with no stable periodic billiard paths.* $(\frac{\pi}{2^{k+1}}, \frac{\pi}{2^{k+1}}, \frac{(2^k-1)\pi}{2^k})$



What about right triangles?

In their article *Periodic billiard trajectories in polygons: generating mechanisms*, Vorobets, Gal′perin, and Stëpin asked:

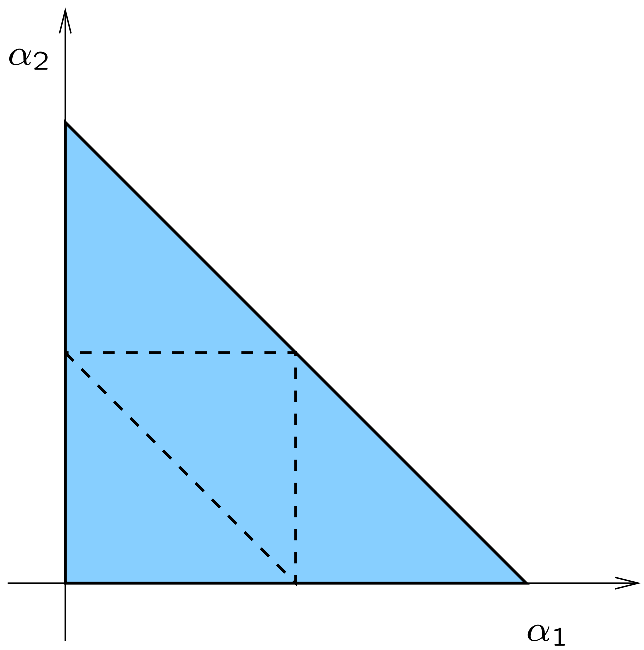
Question. *"Does there exist at least one right-angled triangle containing stable trajectories?"*

Short Answer. *No right triangle has stable periodic billiard paths.*

The long answer

- Parameterize the space of marked triangles $\mathcal{T}$ by the angles of the triangle.

- The right triangles consist of three lines ℓ_1, ℓ_2, and ℓ_3 in this space.



Theorem (Main theorem). *If $\widehat{\gamma}$ is a* **stable** *periodic billiard path in a triangle, then tile($\widehat{\gamma}$) is contained in one of the four components of $\mathcal{T} \setminus (\ell_1 \cup \ell_2 \cup \ell_3)$.*

Vague idea of proof

- You are given a stable periodic billiard path $\widehat{\gamma}$ in a triangle Δ_1.

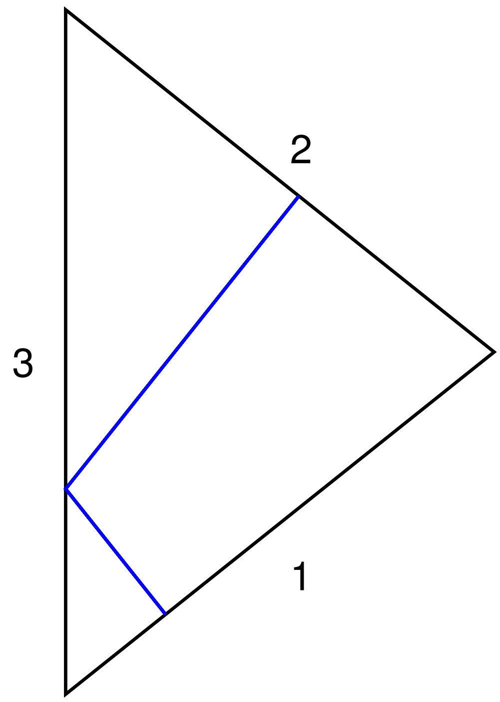
- We define a subset $UF(\widehat{\gamma}) \subset \mathcal{T}$, which is the collection of all triangles Δ where we can find certain "topological obstructions" to the existence of a periodic billiard path with symbolic dynamics $s_{\widehat{\gamma}}$.

Theorem. *$UF(\widehat{\gamma})$ is a finite union of lines in $\mathcal{T}$.*

Theorem (Bounding box). *tile$(\widehat{\gamma})$ lies in one component of $\mathcal{T} \setminus UF(\widehat{\gamma})$.*

Lemma. *Every right triangle lies in $UF(\widehat{\gamma})$.*

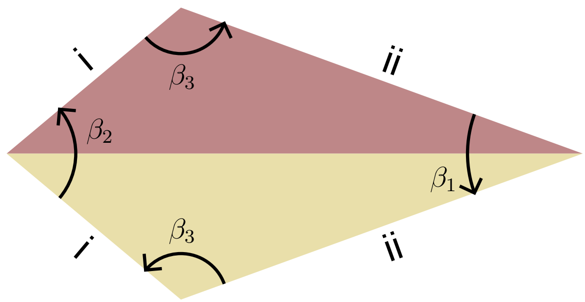
What about unstable periodic billiard paths?



By similar triangles, this periodic billiard path can only exist in isosceles triangles. In fact, it exists in all isosceles triangles.

A deeper look

Given a triangle Δ we can build a Euclidean cone surface $\mathcal{D}_\Delta$ by doubling the triangle across its boundary. Let Σ denote the collection of cone singularities on $\mathcal{D}_\Delta$.



- The billiard flow on Δ lifts to the geodesic flow on $\mathcal{D}_\Delta \setminus \Sigma$.

- Loops invariant under the billiard flow correspond to loops invariant under the geodesic flow on $T_1(\mathcal{D}_\Delta \setminus \Sigma)$.

A trivial obstruction

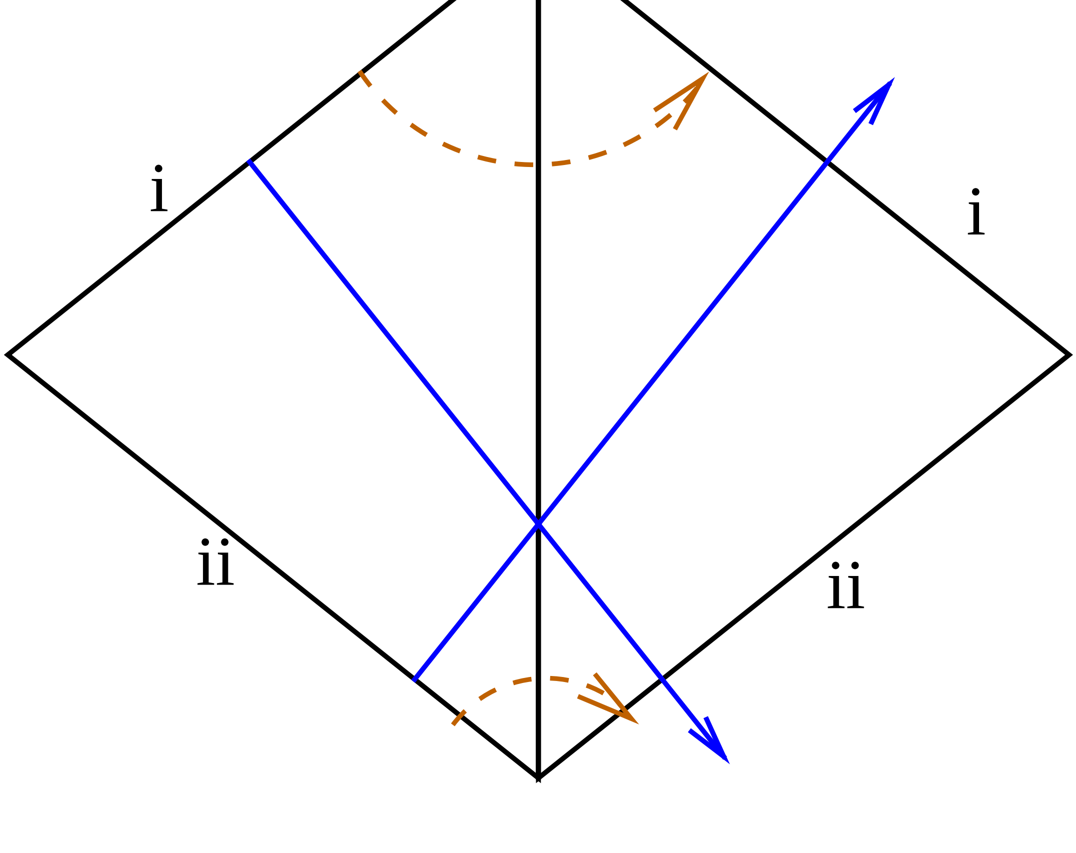
The closed 1-form $d\theta$ on $T_1\mathbb{R}^2$ is invariant under the action of $Isom_+(\mathbb{R}^2)$.

It pulls back to closed 1-form on the unit tangent bundle of any locally Euclidean surface.

A homological obstruction: If the homology class $x \in H_1\big(T_1(\mathcal{D}_\Delta \setminus \Sigma), \mathbb{Z}\big)$ contains a loop invariant under the geodesic flow then

$$\int_x d\theta_\Delta = 0$$

This is our example lifted to $\mathcal{D}_\Delta$.



An algebraic interpretation of stability

Theorem. *A periodic billiard path $\widehat{\gamma}$ in Δ is stable iff the corresponding loop γ is null-homologous on $T_1(\mathcal{D}_\Delta \setminus \Sigma)$.*

Theorem. *If $\widehat{\gamma}$ is unstable, then tile($\widehat{\gamma}$) is contained in the* **rational line**

$$\{\Delta \in \mathcal{T} \text{ such that } \int_\gamma d\theta_\Delta = 0\}$$

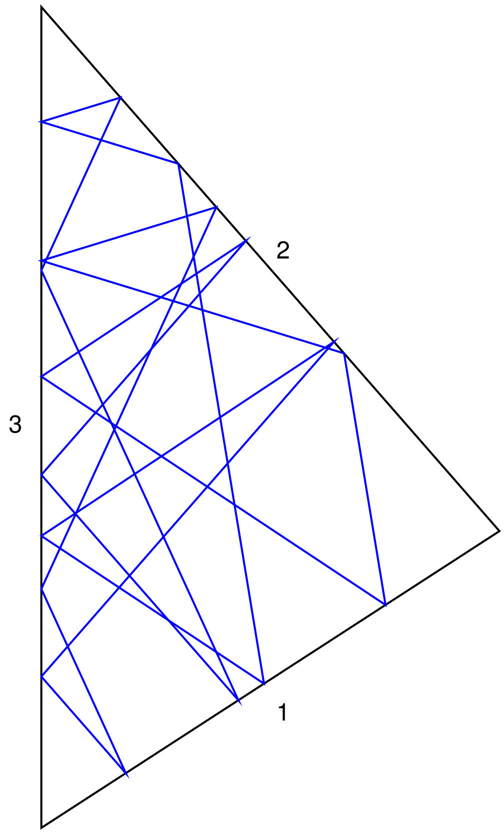
Remark: If the angles of the triangle Δ are $(\alpha_1, \alpha_2, \alpha_3)$ then

$$\int_x d\theta_\Delta = 2n_1\alpha_1 + 2n_2\alpha_2 + 2n_3\alpha_3$$

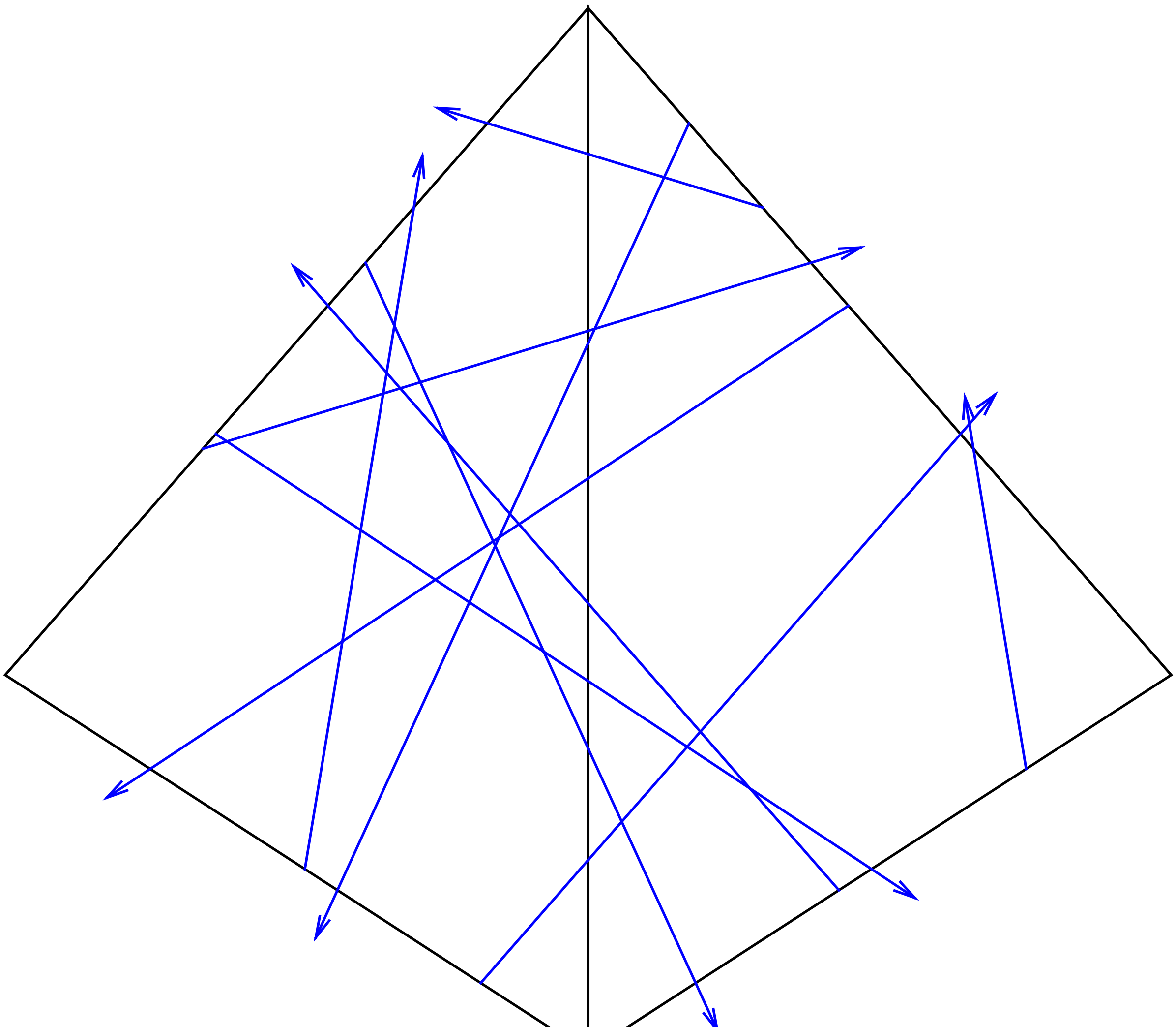
for $n_1, n_2, n_3 \in \mathbb{Z}$ depending on the homology class $x \in H_1\big(T_1(\mathcal{D}_\Delta \setminus \Sigma)\big)$.

The argument in action

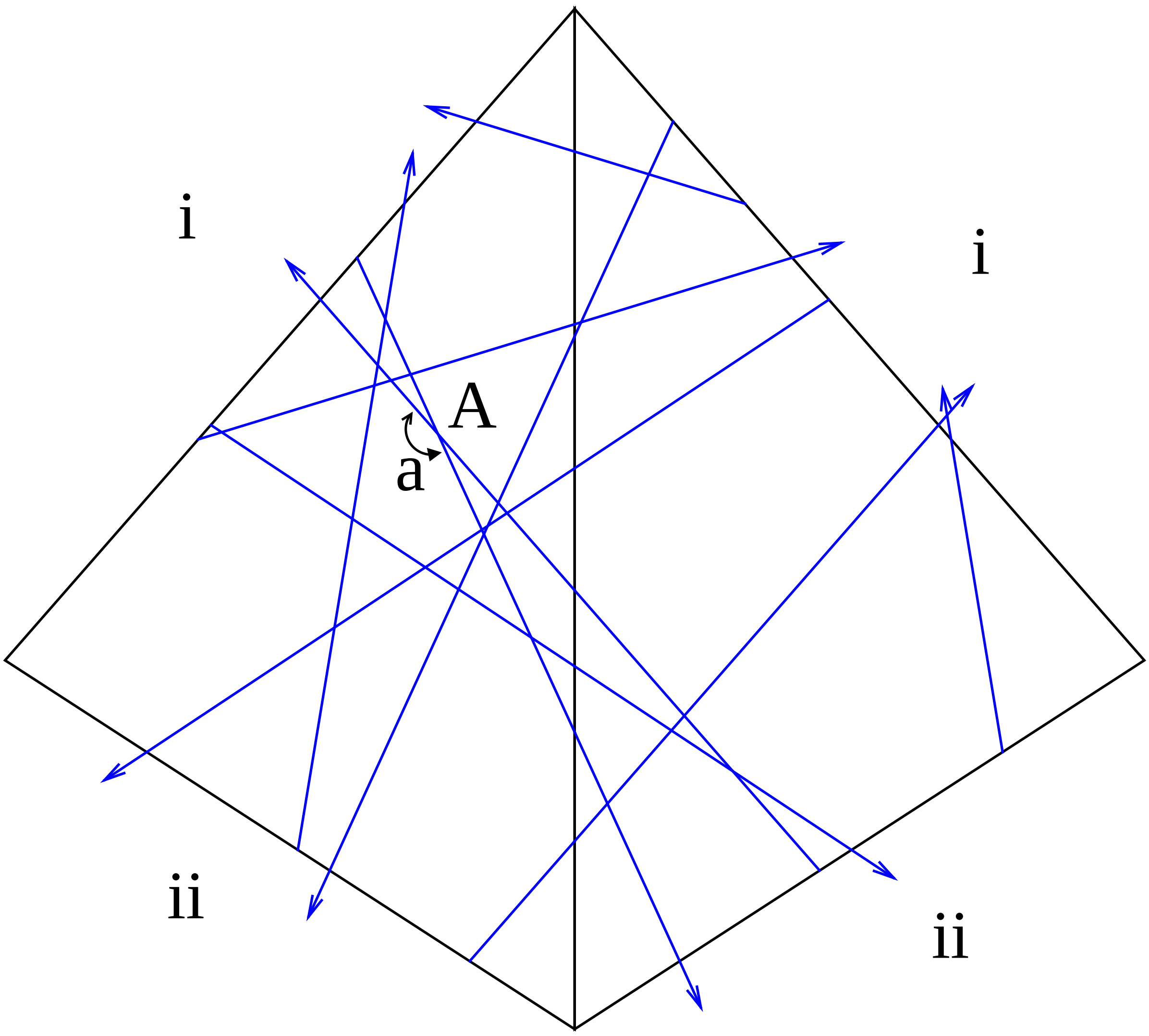
Here is a **stable** periodic billiard path in a slightly acute triangle.

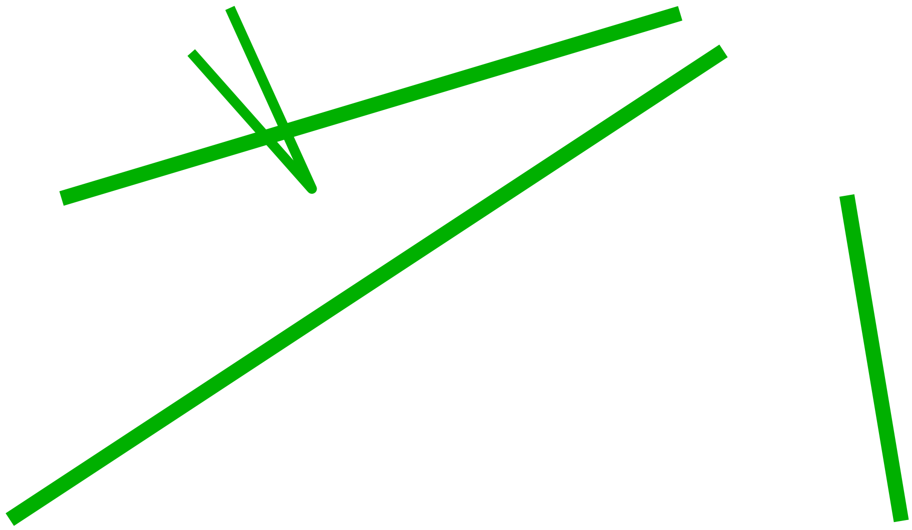


Let's prove that a periodic billiard path with the same symbolic dynamics can not appear in a right or obtuse triangle.



The proof follows from the “general principle” that intersections between geodesics on locally Euclidean surfaces are “essential.”

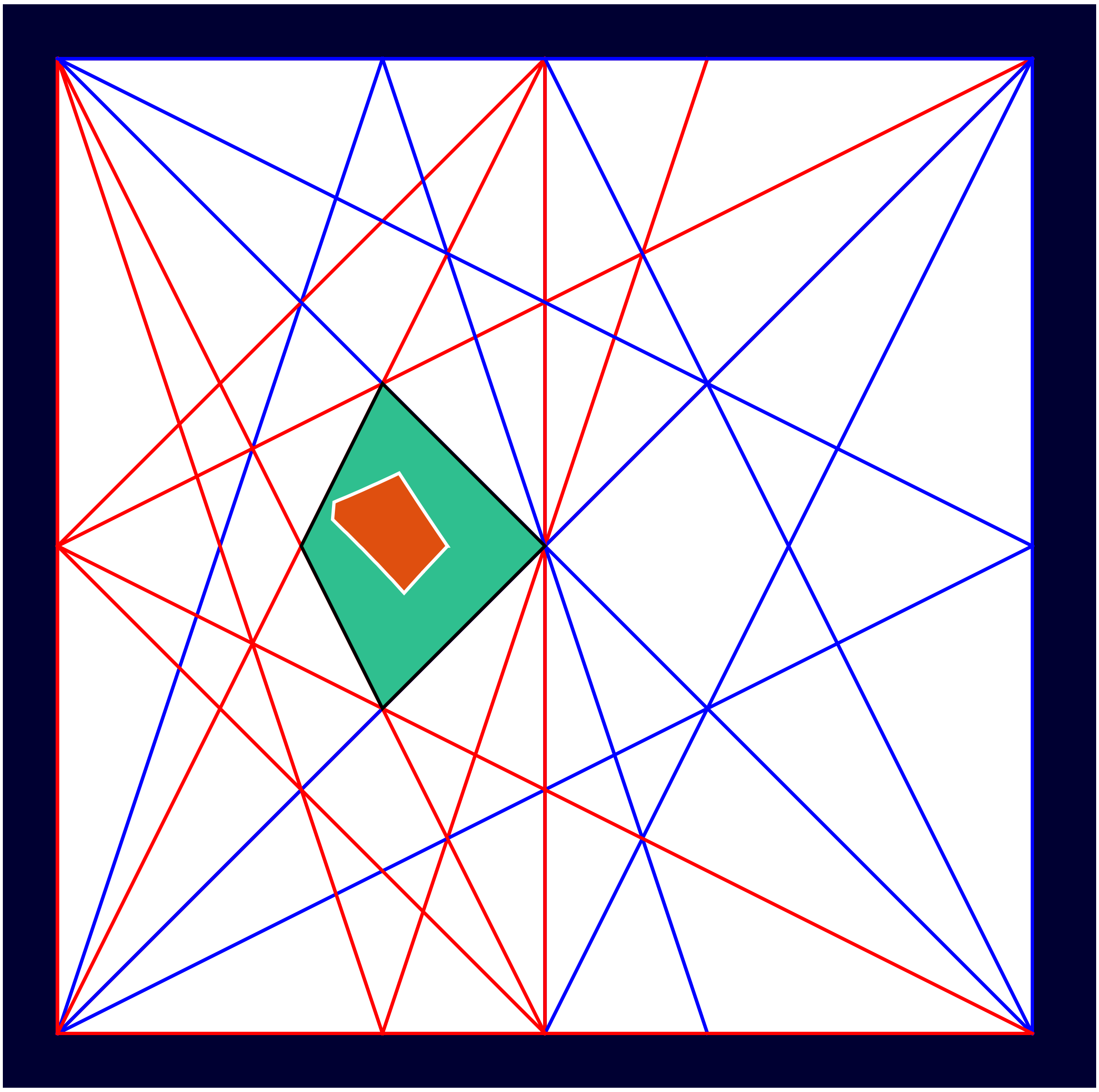




$$a = -\int_\eta d\theta_\Delta = 2\alpha_3$$

For all $\Delta \in \mathit{tile}(\widehat{\gamma})$, $0 < -\int_\eta d\theta_\Delta < \pi$

Iterating over all intersections gives a convex bounding box for the tile.



Let γ be a loop which is invariant under the geodesic flow in $T_1(\mathcal{D}_{\Delta_1} \setminus \Sigma)$.

For each self-intersection of $p(\gamma)$, we get a surgered loop η on $T_1(\mathcal{D}_{\Delta_1} \setminus \Sigma)$ and an obstruction to the existence of a geodesic flow invariant loop in the homotopy class $[\gamma]$ on Δ_2. Namely, we need

$$0 < \int_\eta d\theta_{\Delta_2} < \pi$$

The proof of the main theorem, that stable periodic billiard paths in acute and obtuse triangles never have the same symbolic dynamics, would follow if we could prove that:

"Let γ be any null-homologous geodesic on $T_1(\mathcal{D}_{\Delta_1} \setminus \Sigma)$ for any triangle Δ_1. For all right triangles Δ_2, we can find an intersection of $p(\gamma)$ so that the surgered curve η satisfies

$$\int_\eta d\theta_{\Delta_2} \equiv 0 \pmod{\pi}\text{"}$$

What happens when

$$\int_\eta \mathbf{d}\theta_{\Delta_2} \equiv 0 \pmod{\pi}?$$

The minimal translation surface cover, MT_Δ, of $\mathcal{D}_\Delta \setminus \Sigma$ is the cover chosen so that a loop ζ on $T_1(\mathcal{D}_\Delta \setminus \Sigma)$ lifts to $T_1(MT_\Delta \setminus \Sigma)$ iff

$$\int_\zeta d\theta \equiv 0 \pmod{2\pi}$$

Translation Surfaces

Definition. *A translation surface TS is a Euclidean cone surface, where all cone angles are integer multiples of* 2π*. We also allow infinite cone angles.*

All cone surfaces can be built out of pieces of $\mathbb{R}^2$ glued together by translations.

So the notion of direction on $\mathbb{R}^2$ (the map $\theta : T_1\mathbb{R}^2 \to \mathbb{R}/2\pi\mathbb{Z}$) pulls back to a translation surface.

Translation Surface Covers

- The universal cover $\widetilde{\mathcal{D}_\triangle \setminus \Sigma}$
- The **universal abelian cover**

  $AC_\triangle = \widetilde{\mathcal{D}_\triangle \setminus \Sigma} / [\pi_1(\mathcal{D}_\triangle \setminus \Sigma), \pi_1(\mathcal{D}_\triangle \setminus \Sigma)]$

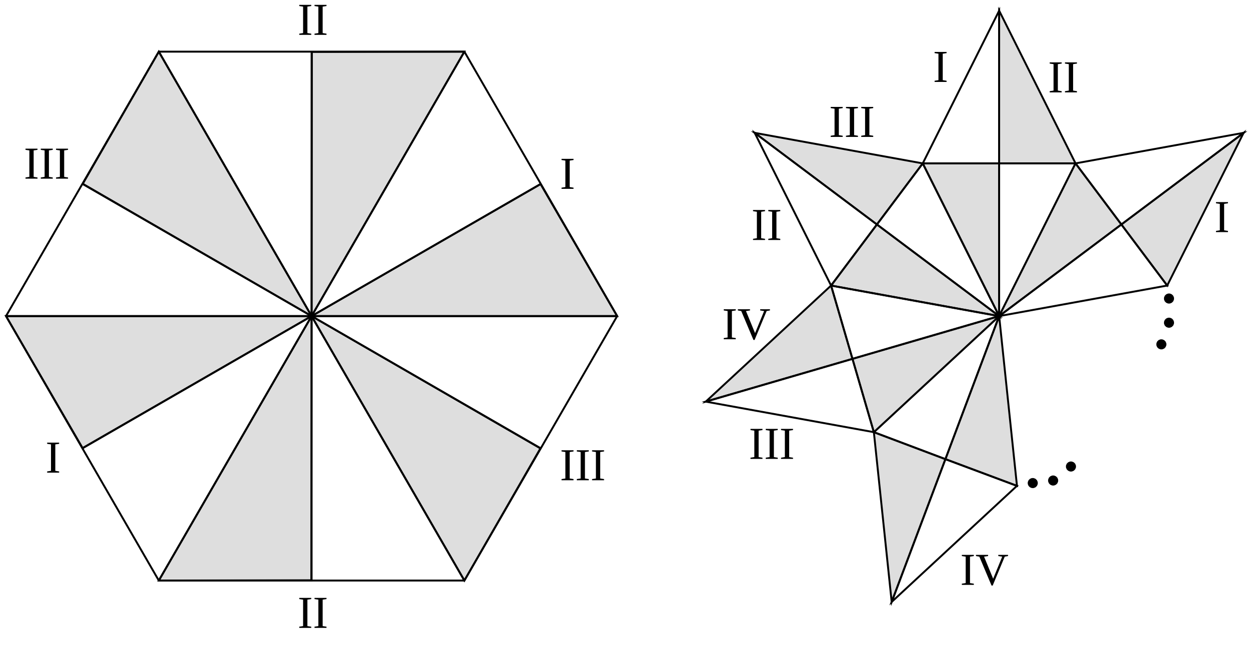
  It has a cover automorphism group isomorphic to $H_1(\mathcal{D}_\triangle, \mathbb{Z}) = \mathbb{Z}^2$.

- The **minimal translation surface** cover (aka the **invariant surface**). It can be built as $\widetilde{\mathcal{D}_\triangle \setminus \Sigma}$ modulo those elements of $\pi_1(\mathcal{D}_\triangle \setminus \Sigma)$ whose holonomy is a translation.

We have the following sequence of branched covers:

$$\widetilde{\mathcal{D}_\triangle \setminus \Sigma} \to AC_\triangle \to MT_\triangle \to \mathcal{D}_\triangle$$

Some minimal translation surface covers



Geodesics on Translation Surfaces

Because the direction map on a translation surface ($\theta : T_1 TS \to \mathbb{R}/2\pi\mathbb{Z}$) is invariant under the geodesic flow, geodesics travel in a fixed direction.

Furthermore,

TO-1: A geodesic has no self intersections.

TO-2: A pair of distinct geodesics traveling in the same direction never intersect.

TO-3: The absolute value of the algebraic intersection number between two geodesics equals the geometric intersection number.

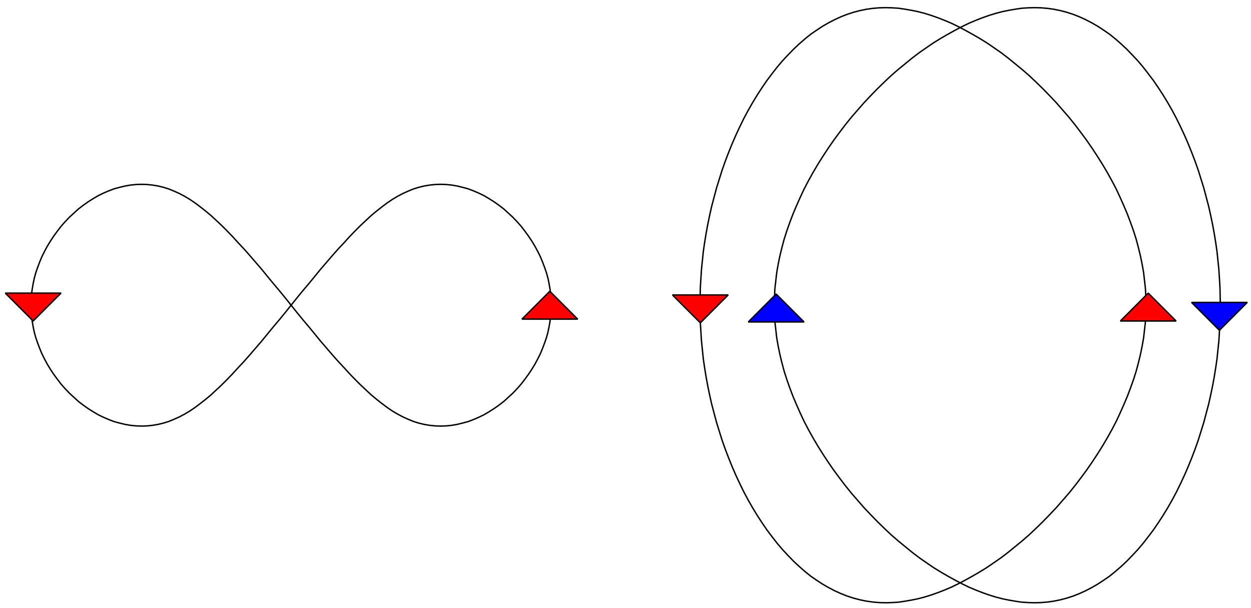
The weaker obstruction

Recall: We had a triangle Δ_1 and a null-homologous loop γ in $T_1(\mathcal{D}_{\Delta_1} \smallsetminus \Sigma)$ invariant under the geodesic flow. There was a self-intersection of $p(\gamma)$ making a figure-8. And a curve η in $T_1(\mathcal{D}_{\Delta_1} \smallsetminus \Sigma)$ projecting to one loop of that figure-8. We saw that on *tile*(γ),

$$0 < \int_{[\![\eta]\!]} d\theta_\Delta < \pi$$

If Δ_2 is a triangle with $\int_{[\![\eta]\!]} d\theta_{\Delta_2} = 0$, then we get a topological obstruction to the existence of a geodesic on $\mathcal{D}_{\Delta_2}$ in the homotopy class $[\gamma]$. Namely, the lift of $[p(\gamma)]$ to MT_{Δ_2} has an "essential intersection" (because the whole figure-8 lifts). This violates *TO-1*.

If Δ_2 is a triangle with $\int_{[\![\eta]\!]} d\theta_{\Delta_2} = \pi$, then a double cover of $[\eta]$ lifts.

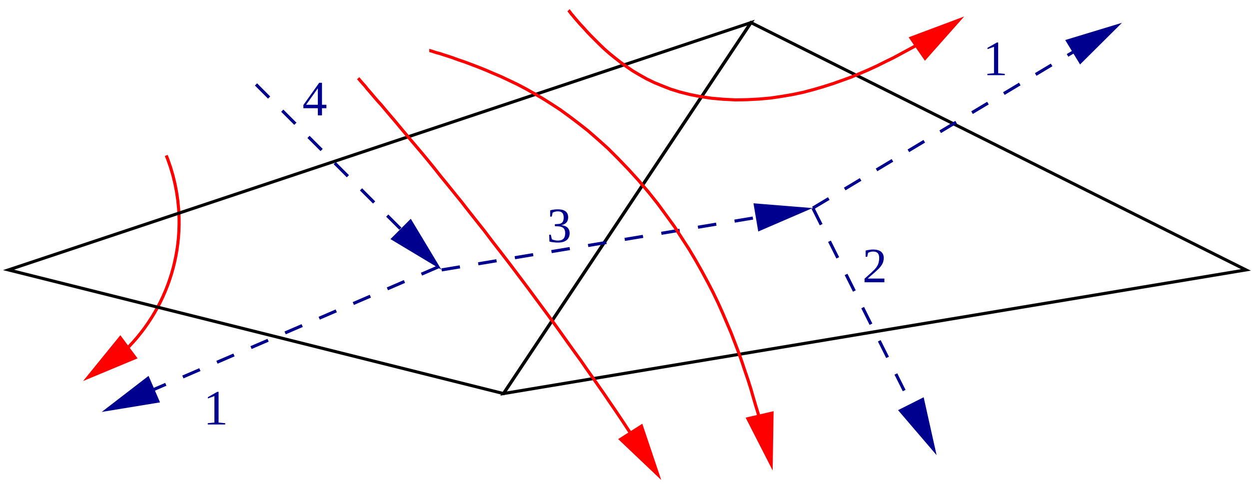


We can lift $[p(\gamma)]$ to two curves $[g_1]$ and $[g_2]$ on MT_{Δ_2} which differ by a Deck transformation, ρ, which rotates by π. They have two "essential intersections" with opposite algebraic signs. We get violations of both *TO-2* and *TO-3*.

Train tracks and the more general topological obstruction

Proposition. *Let* $x = \{[c_1], \ldots, [c_k]\}$ *be a collection homotopy classes on a translation surface TS which can be realized by geodesics all traveling in the same direction. This collection of homotopy classes is uniquely determined by its homology class* $[\![x]\!] = [\![c_1]\!] + \ldots + [\![c_k]\!] \in H_1(TS)$*.*

Proof: In our cases, we have a triangulation of our translation surface surface by saddle connections.

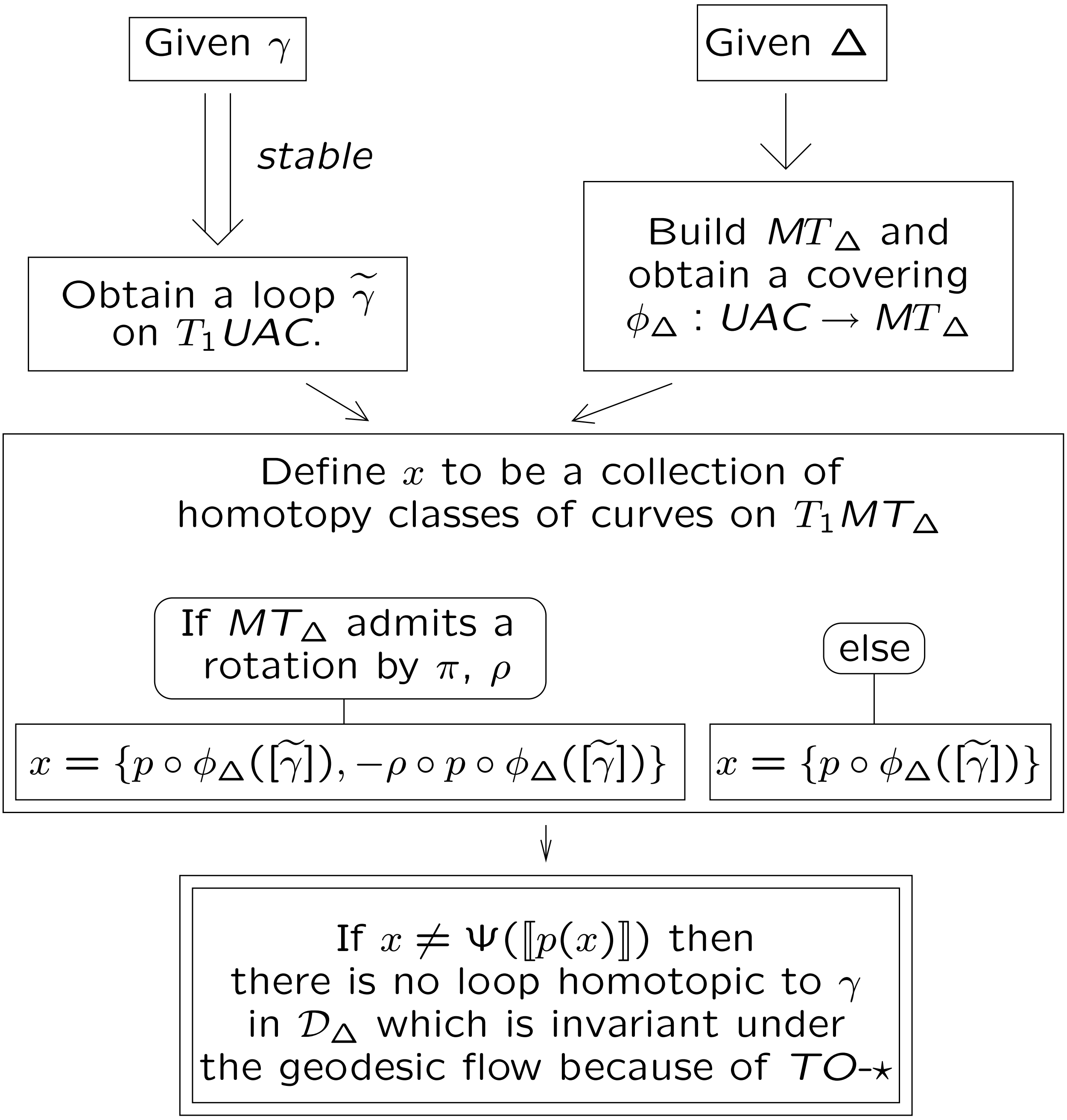


This train track argument gives us a map Ψ from $H_1(\mathit{TS})$ to the set of all finite collections of homotopy classes of loops on $T_1\mathit{TS}$ which can be realized as curves with no violations of *TO-1*, *TO-2*, or *TO-3*.

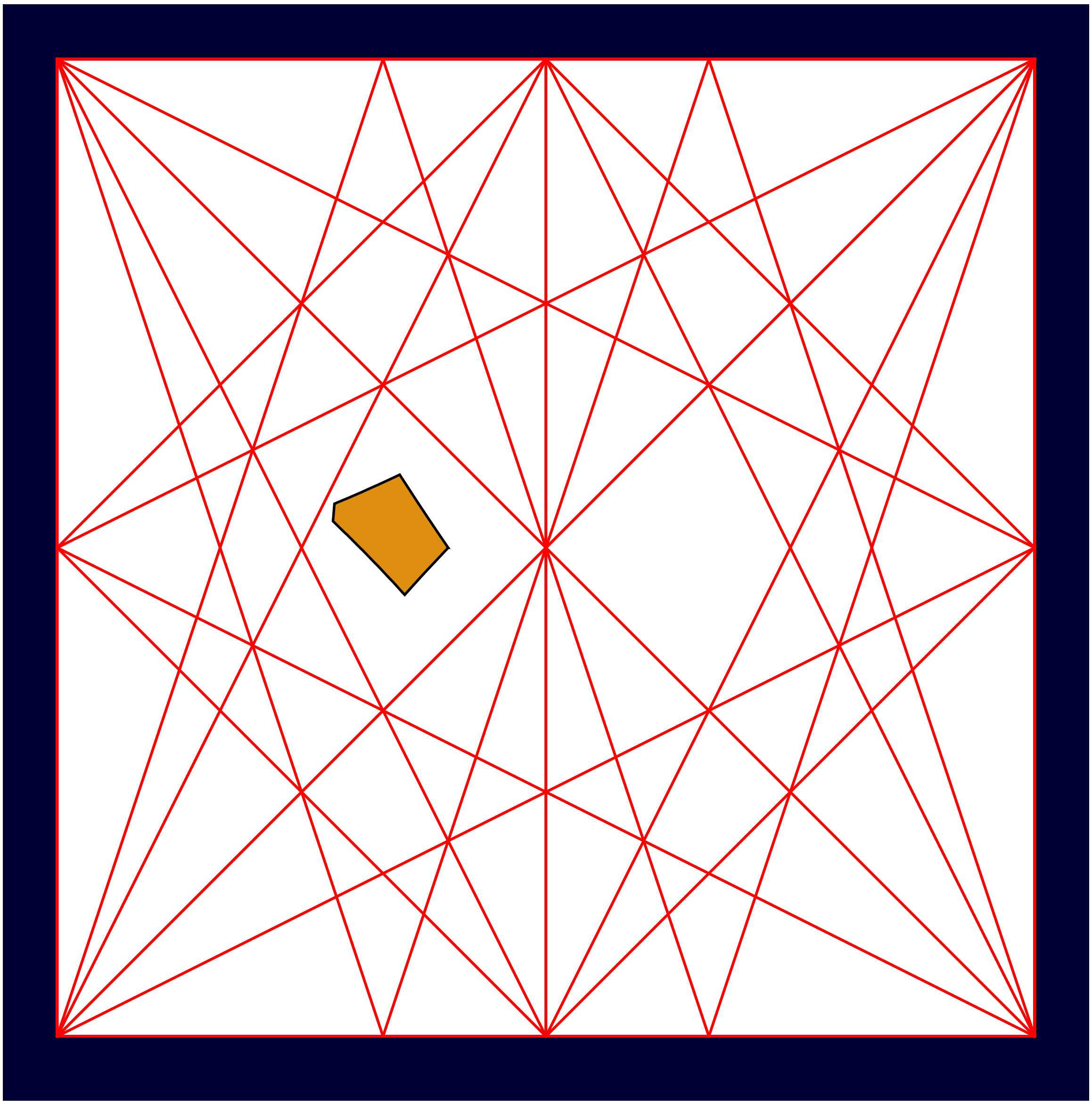
The unfriendly set

Given a null homologous γ on $T_1(\mathcal{D}_{\Delta_1} \setminus \Sigma)$ invariant under the geodesic flow, we define the **unfriendly set** $\mathit{UF}(\gamma) \subset \mathcal{T}$ to be the collection of triangles Δ with topological obstructions to the existence of a geodesic flow invariant loop in $[\gamma]$ on $T_1(\mathcal{D}_{\Delta} \setminus \Sigma)$.

The unfriendly set



An example unfriendly set



Some theorems

Theorem 1. *For $\widehat{\gamma}$ a stable periodic billiard path, The unfriendly set UF($\widehat{\gamma}$) is a finite union of rational lines.*

Each violation of *TO*-$\star$ is equivalent to some detecting curve η on $T_1(s_\Delta \smallsetminus \Sigma)$ (or its double) lifting to MT_Δ.

Theorem 2 (Bounding Box). *tile($\widehat{\gamma}$) is contained in at most one component of $\mathcal{T} \smallsetminus$ UF(γ).*

Theorem 3. *The right triangle lines are contained in the unfriendly set.*