Motivating open question: Which triangles have closed (a.k.a. periodic) billiard paths?

- Every acute triangle has a periodic billiard path, called the Fagnano curve. [Fagnano, 1775]

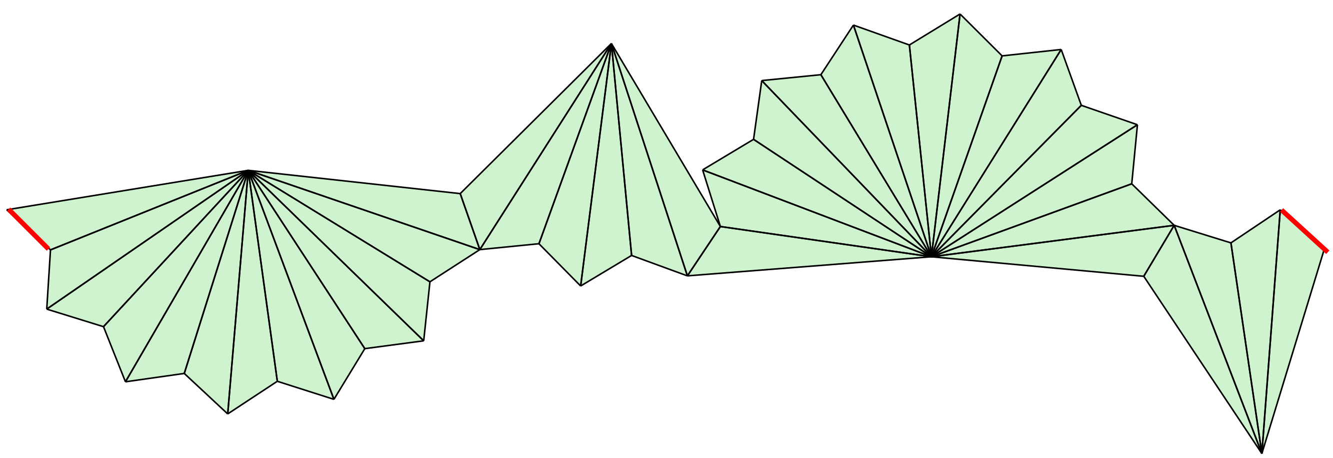
- Every right triangle has a periodic billiard path.

- Every rational polygon has a periodic billiard path. [Masur]

- Halbeisen and Hungerbuhler found some infinite families of periodic billiard paths in obtuse triangles.

The *orbit-type* of a periodic billiard path is the (bi-infinite) sequence of edges it hits.

Given some periodic sequence of edges and a triangle, how do we tell if there is a periodic billiard path in this triangle realizing the sequence as its orbit-type?

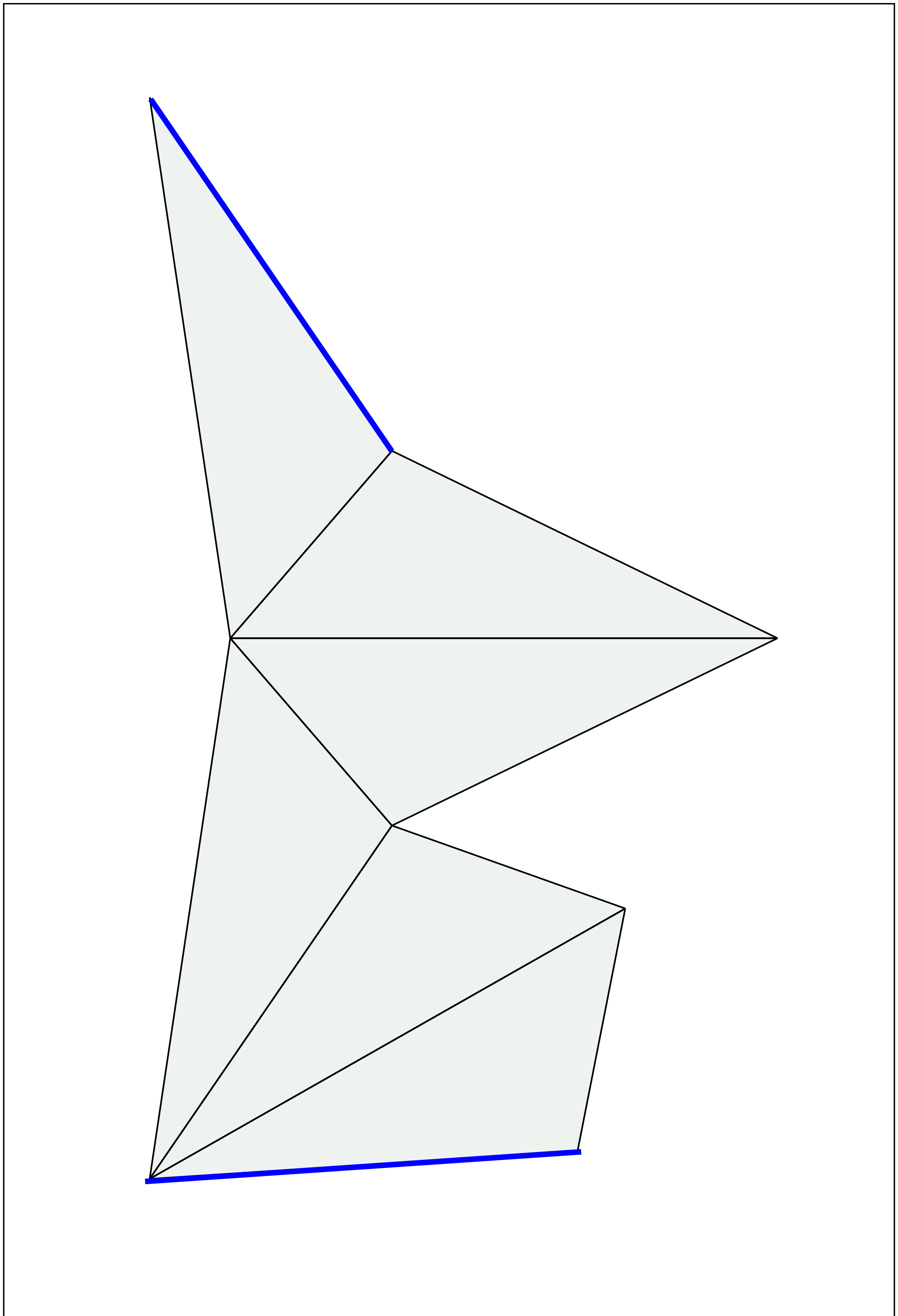


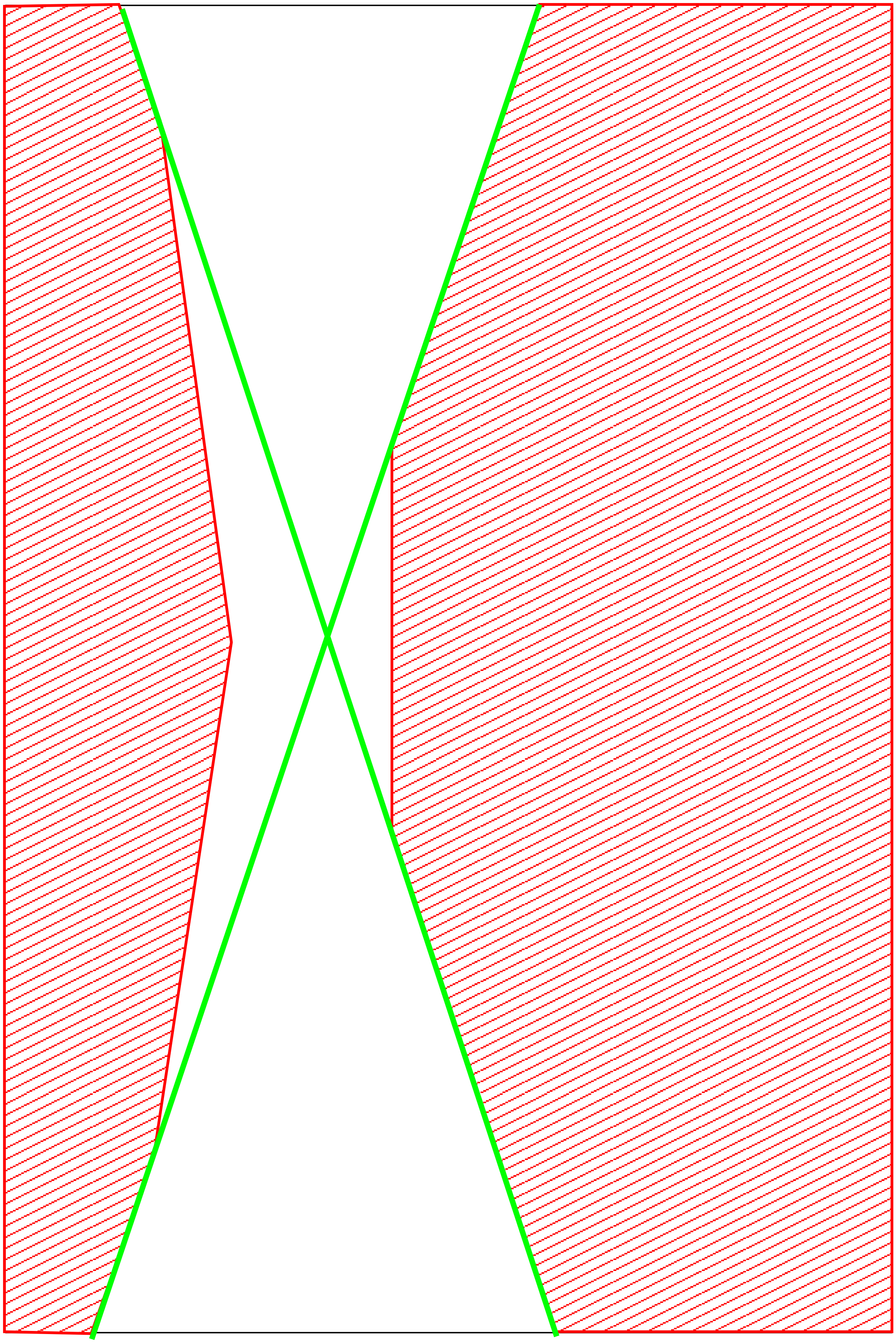
Searching for Billiard Paths

The orbit-types of all finite billiard paths in a fixed triangle Δ have the structure of a tree. (It is a subtree of the infinite trivalent tree.)

Theorem (Cassaigne-Hubert-Troubetzkoy). *In a rational polygon, the number of orbit-types of length n which are realized by finite billiard paths is bounded above and below by cubic polynomials.*

A simple search algorithm would be to traverse this tree. Whenever you see an unfolding where the first and last edges are parallel, check to see if you can construct a billiard path.





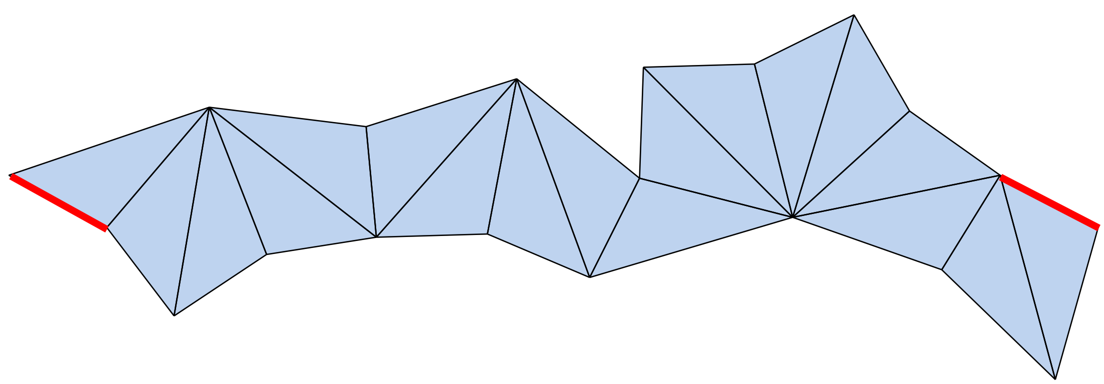
The space of all marked triangles modulo congruence, $\mathcal{T}$, is given the structure of a 2-simplex by using angles as coordinates.

If we fix the orbit-type w, then Tile(w) is the set of triangles with periodic billiard paths with orbit-type w.

Assuming Tile(w) is not empty, the tiles come in one of two flavors depending on w:

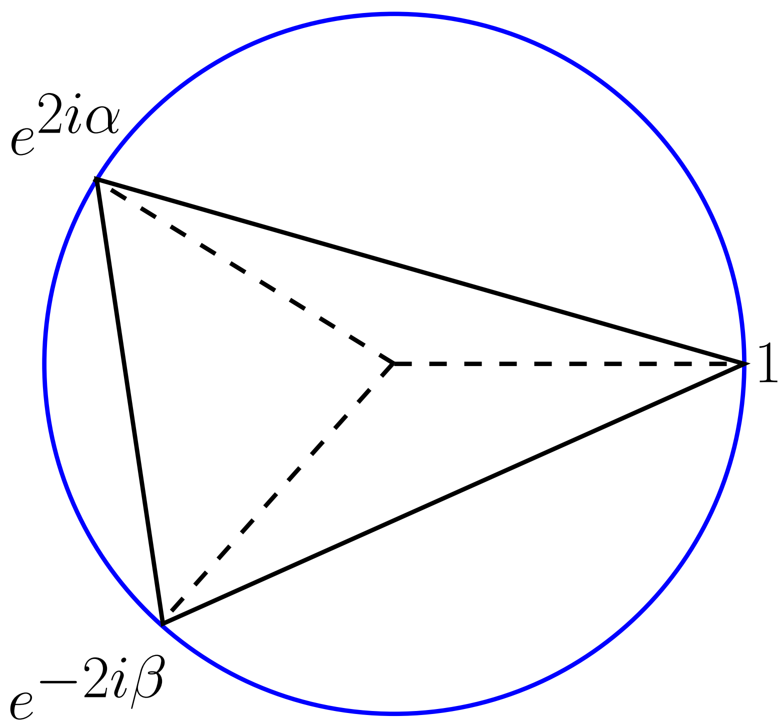
1. Tile(w) is an open subset of $\mathcal{T}$. In this case w is called *stable*.

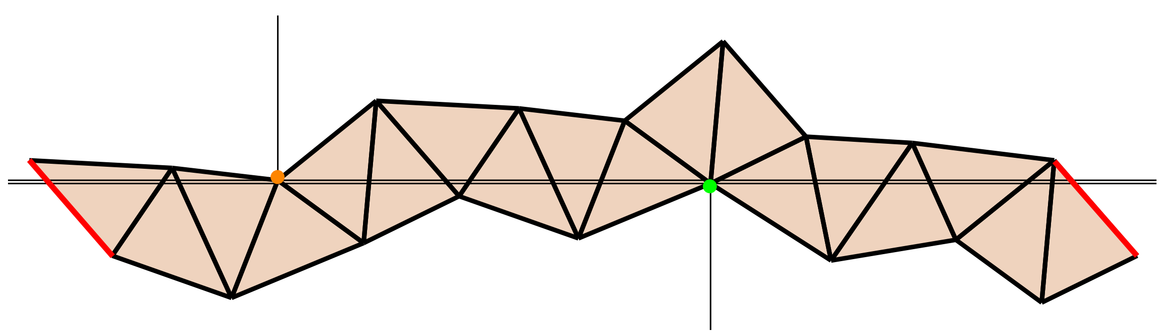
2. Tile(w) is an open subset of some rational line in $\mathcal{T}$. We call w is called *unstable*.



Plotting Tiles

If α and β are two angles of the triangle, then all the vertices of the unfolding can be arranged to lie in the ring $\mathbb{Z}[e^{\pm 2i\alpha}, e^{\pm 2i\beta}]$.





Theorem (Schwartz). *If the largest angle of a triangles is less than 100 degrees, then the triangle has a periodic billiard path.*

Related open question: Which triangles have stable periodic billiard paths?

- If the largest angle of a triangle is between 90 and 100 degrees, then the triangle has a stable periodic billiard path. [Schwartz]

- Right triangles do not have stable periodic billiard paths. [H]

- Isosceles triangles with angles $(\frac{\pi}{2^n}, \frac{\pi}{2^n}, \bigstar)$ do not. [H]

- Isosceles triangles with angles not equal to $(\frac{\pi}{2n}, \frac{\pi}{2n}, \frac{(n-1)\pi}{n})$ do. [H]

- We suspect the remaining isosceles triangles do have stable periodic billiard paths.

In fact, some of the triangles which don't have stable billiard paths have an even worse property.

Theorem (Schwartz). *There is a sequence of triangles* Δ_i *approaching the 30-60-90 triangle with the property that there are no periodic billiard paths of length less than* i *in* Δ_i*.*

Corollary (Schwartz). *No neighborhood of the 30-60-90 triangle is covered by finitely many tiles.*

We expect that these statements are also true for the $(\frac{\pi}{2^n}, \frac{\pi}{2^n}, \bigstar)$ isosceles triangles.

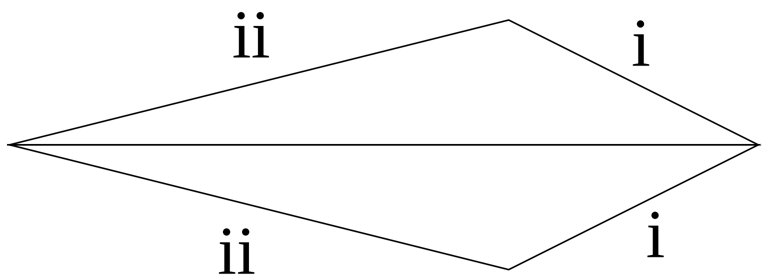
Topology and Irrational Billiards

Another interesting question: Given an orbit-type (a periodic sequence of edges) which triangles have a periodic billiard path with that orbit-type?

It turns out we can extract some partial answers to this question from topological information alone. For example:

Theorem (H). *No acute triangle has a stable periodic billiard path with the same orbit type as a stable periodic billiard path in an obtuse triangle.*

We can construct a Euclidean cone structure on the 3-punctured sphere by gluing a triangle Δ to an orientation reversed copy of Δ.

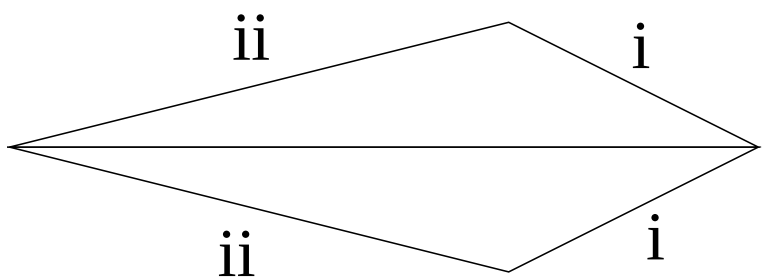


There is a folding map $S_\Delta \to \Delta$ which takes geodesics to billiard paths.

A periodic billiard path on Δ lifts unless is has odd period. If it has odd period then its double cover lifts.

Theorem. *If γ is a lift of stable periodic billiard path to S_Δ, then it is null homologous.*

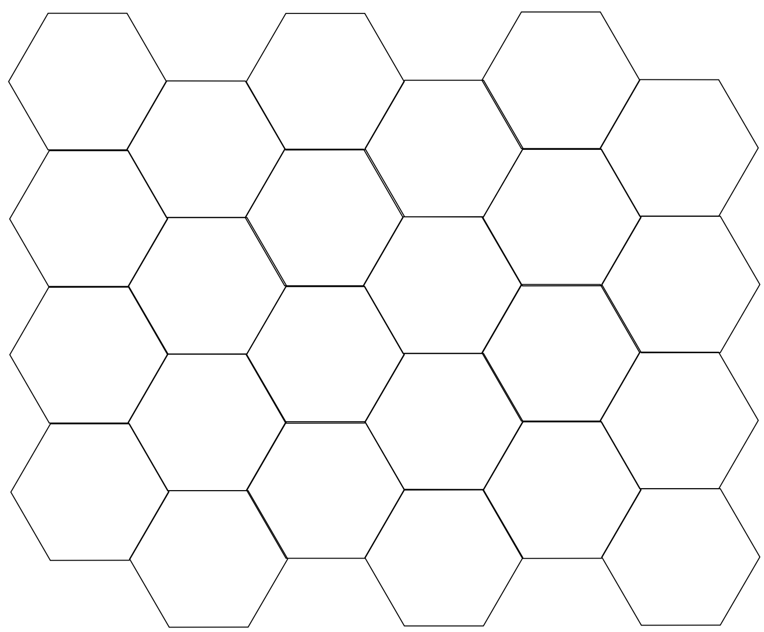
$$\begin{array}{ccc} \pi_1(S_\Delta) & \xrightarrow{\text{hol}} & \text{Isom}_+(\mathbb{R}^2) \\ \downarrow & & \downarrow \\ H_1(S_\Delta, \mathbb{Z}) & \xrightarrow{\text{hol}_{\text{ab}}} & S^1 \end{array}$$

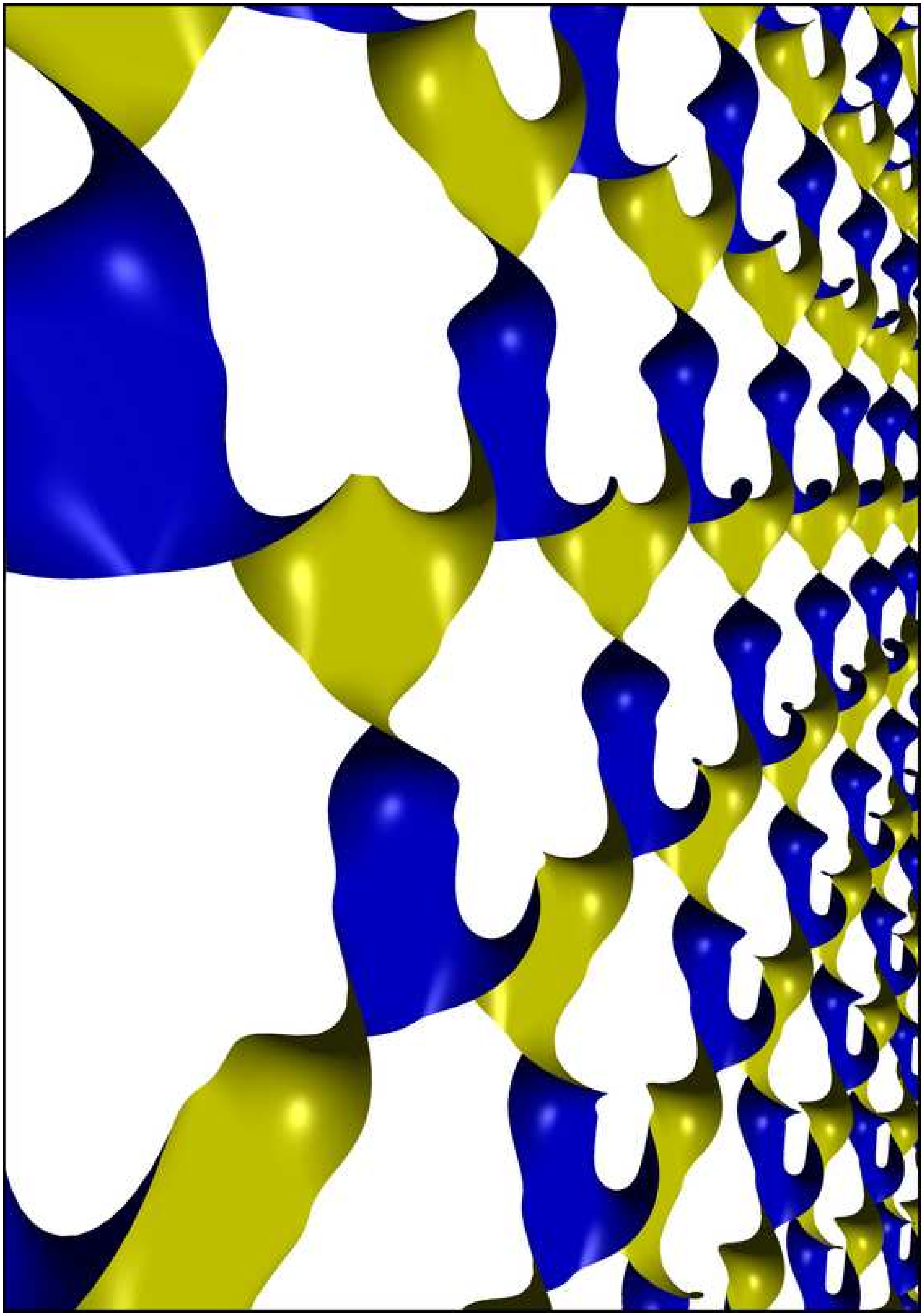


So, stable paths lift to the universal abelian cover of S_Δ.

$$\mathrm{UAC}(S_\Delta) = \widetilde{S}_\Delta / [\pi_1(S_\Delta), \pi_1(S_\Delta)]$$

S_Δ is homotopy equivalent to the Θ-graph, so UAC(S_Δ) and UAC(Θ) are homotopy equivalent equivalent.





The universal abelian cover of S_Δ is a translation surface. That is, it can be constructed from pieces of $\mathbb{R}^2$ that are glued together by translations.

In general, given a Euclidean cone surface S we can construct it's universal translation surface covering, UTS(S). Let $K \subset \pi_1(S)$ be the subgroup of elements with translational holonomy.

$$\mathrm{UTS}(S) = \widetilde{S}/K$$

Every translation surface covering of S covers UTS(S). In particular UAC(S) covers UTS(S).

- For generic triangles UTS(S_Δ) $=$ UAC(S_Δ).

- For rational triangles UTS(S_Δ) has finitely many punctures, and the deck group of the covering UAC(S_Δ) $\to$ UTS(S_Δ) is $\mathbb{Z} \oplus \mathbb{Z}$.

- Otherwise Δ in generic in some rational line, and the deck group is $\mathbb{Z}$.

The triangulation of S_Δ by two copies of Δ pulls back to a triangulation of UTS(S_Δ). Closed geodesics on UTS(S_Δ) are

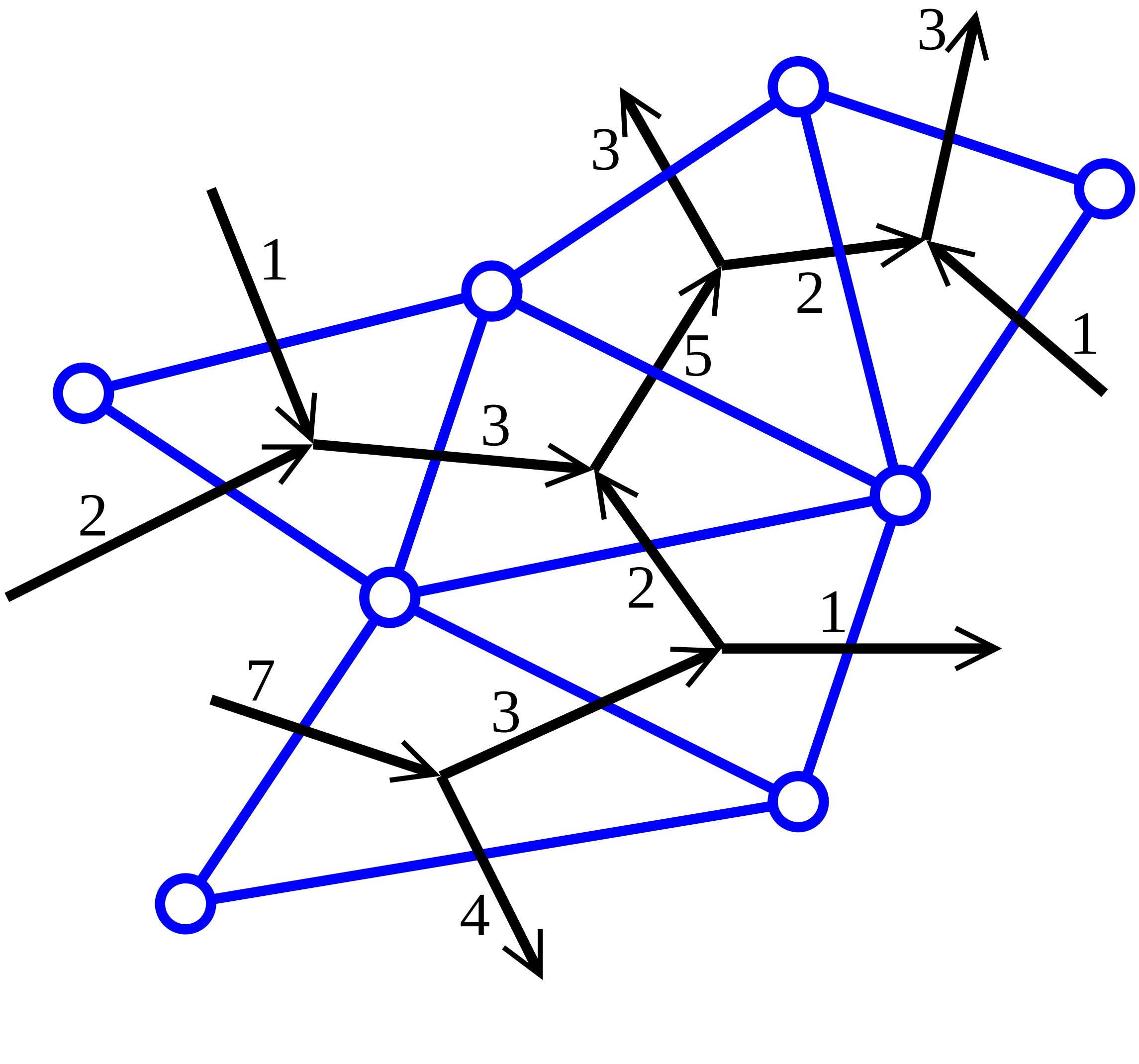
1. simple and

2. intersect each edge of the triangulation in at most one direction.

Call a topological path on translation surface with such a triangulation that satisfies 1 and 2 *billiard-like*.

If there is a rotation by 180 degrees that preserves the translation surface, we require slightly more.

Proposition. *If S is a translation surface triangulated by edges connecting singularities, then the restriction of the map $\pi : \pi_1(S) \to H_1(S, \mathbb{Z})$ to the set of billiard-like paths is injective.*



Theorem (H). *Suppose* $\Delta \in \mathcal{T}$ *is generic in some rational line* ℓ *and* $\gamma \in \pi_1(UAC(S_\Delta))$ *is billiard-like on* $UAC(S_\Delta)$*. If* γ*'s projection to* $UTS(S_\Delta)$ *fails to be billiard-like, then the Tile(*γ*) lies in at most one component of* $\mathcal{T} \setminus \ell$*. The component can be determined from information about this failure.*

Theorem (H). *If* Δ *is a right triangle, no billiard-like path on* $UTS(\Delta)$ *lifts to* $UAC(\Delta)$*.*

In general given an orbit-type w, the first theorem allows us to construct a convex rational bounding box B_w for the tile from topological information alone.

Question: Does every irrational triangle Δ lie in some B_w?